THE NUMBER SYSTEM

OF ARITHMETIC AND ALGEBRA.

D.K.PICKEN.

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BY

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To
R. A. HERMAN

PREFACE.

The object of this little book is to give a more or less complete account of the essentials of the Number-system of Arithmetic and Algebra—from its origin, in the intuitive number-ideas of the child, to its final development, in the Complex Number work of mathematical theory.

Some twenty years of university teaching, and some six or seven years' chairmanship of Public Examination Boards in this university, have convinced the writer of the need, from these two different points of view, for such a book. Part I. fulfils an undertaking to provide the kind of help that seems to be necessary for teachers, and their pupils, in school work—besides being the necessary foundation of the wider plan.

The thesis of Part I. is that the principles of Arithmetic-and-Algebra constitute a scheme which is one and indivisible: that algebraic principles are necessary for the understanding of the "rules" of Arithmetic; and that, on the other hand, these principles of Algebra can be properly understood only when applied at every point to the arithmetical facts which form their essential subject-matter.

An essential feature of the treatment of the subject is the special place given to the Natural Numbers—so as to retain them in pristine simplicity as *a reservoir of number-ideas*, upon which to draw freely in the subsequent general mathematical development. This implies a definite distinction between the Natural Numbers and the Positive Integers—which amounts, in effect, to this, that the latter system is *not* subject to the severe restrictions which are necessary to the simplicity of the Natural Number system (and are therefore regarded as characteristic of that system). It also implies that in mathematical practice (as distinct from the arithmetic of elementary "counting") the Natural Numbers are superseded by the Positive Integers.

The common thread of theoretical argument running through the book is the establishing in Mathematics of a system of numbers, subject, *without restriction*, to generalised operations based upon the seven operational forms of the original operation of Addition—which is inherent in the Natural Number system.

But the practical principle, of the relation of Number to Ratio and Measurement, is regarded as only secondary in importance; and each of these two principles is used to strengthen the effect of the other. There is need for a fresh recognition of the place of Number in Natural Philosophy—as something about which we learn from sources outside ourselves.

Part II. is necessarily much more difficult than Part I.; but great pains have been taken to keep the difficulty at the irreducible minimum, consistent with sufficient accuracy of treatment. It is hoped that a presentation of facts as to the Real Numbers and the Complex Numbers has been given, which will be of value to students of Physics, Engineering, Philosophy, and Education, as well as to students of Pure Mathematics.

Mention may perhaps be made of the use to which the Decimal Numbers are put, to bridge the chasm between the Rational Numbers and the Irrational Numbers; of the fact that it has been found possible, in a legitimate way, to avoid defining one number as a "class" of other numbers; of the restriction to "one-valuedness" rigidly imposed on Involution (and its inverses) in the Chapter on Real Numbers, so as to postpone the complication of "many-valuedness" to its proper context in the following Chapter; and of the discussion given of the relation of the Complex Numbers to the facts of Plane Geometry.

The objective of the book has been interpreted as excluding details of a highly-specialised mathematical character. Hence proofs of this character have been omitted. The omission could be made good in an Appendix, should that appear at some future time desirable.

The four Appendices deal with matters, essential to the plan of the book, for which it did not seem advisable to interrupt the general argument. The most debatable of these is Appendix II., which discusses a question the writer believes to be of great importance to mathematical practice. Appendix III. deals exhaustively with the exceptional cases of the operational processes, mainly associated with the special questions of "nought" and "infinity." Appendix IV. gives a very simple treatment of a very important fundamental topic.

It will be obvious why no sets of examples have been included.

Besides the special obligation expressed—all too inadequately—in the dedication, and indebtedness to innumerable other influences, in the literature of the subject, the writer wishes to single out for mention two classic publications which made a great impression upon him at a critical point of his work as a university teacher: Dedekind's Essay on "Continuity and Irrational Numbers," and Lagrange's "Lectures on Elementary Mathematics." This book, however, represents an essential unity of the writer's thought and experience; it is not consciously based, in any specific way, on the work of others.

Thanks are due to Mr. E. O. Hercus, M.Sc., of the Natural Philosophy Department of this university, for painstaking care in reading proofs, and for valuable suggestions as to points in the presentation; also to the printers for their patience in producing work much of which was of an unfamiliar kind.

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PART I.

ELEMENTARY PRINCIPLES
OF ARITHMETIC AND ALGEBRA.

CHAPTER I.

THE NATURAL NUMBERS AND THE OPERATIONS.

1. The source of Number ideas is the system of "Natural Numbers," which we call "one, two, three, · · · · · ·.," and denote by 1, 2, 3, · · · · · · (infinitely extended. See § 10, (i.) and Appendix III.). It is of the utmost importance that these numbers and their properties should be thoroughly understood, as a preliminary and a foundation to the whole science of Pure Mathematics.

2. (i.) The "operation" of **Addition** is inherent in the system of Natural Numbers. We write, for example—

$$3 + 5 = 5 + 3 = 8;$$

and, in general algebraic form, $a + b = b + a = c$; any natural number "values" of a and b giving a corresponding natural number value of c, which is called their "sum." And we have immediately the extension to the addition of more than two numbers, represented by—

$$a + b + c + d + \cdot \cdot \cdot \cdot \cdot \cdot$$

(ii.) Here we come first upon two of the three fundamental types of "law" with which a great part of elementary Arithmetic-and-Algebra is concerned: laws which express the fact that a given set of numbers has a definite "sum"—as follows:—

(1) The *order* of the terms in an addition expression is immaterial to the sum; for example—

$$3 + 5 + 9 = 3 + 9 + 5 = 5 + 3 + 9 = 5 + 9 + 3 = \text{etc.} = 17$$

(2) The sum may be arrived at by grouping the terms into sub-expressions of the same kind, in any way, and adding their sums; for example—

$$3 + 5 + 2 + 9 + 6 = (3 + 5) + (2 + 9) + 6$$
$$= 3 + (5 + 2 + 9) + 6 = \text{etc.}$$

These two general propositions are technically known as the Laws of (1) "Commutation" and (2) "Association" in Addition; and there seems to be no sufficient reason why these technical terms should be any less familiar in Secondary Education than the terms "Addition," "Subtraction," etc. But whether or not the terms are used, it is absolutely essential that the laws themselves should be clearly understood.

3. (i.) **Subtraction** is—and remains throughout Algebra—the operation "inverse" to Addition.

As equivalents to—

$$3 + 5 = 5 + 3 = 8$$

we write $8 - 5 = 3$ and $8 - 3 = 5$; or, in general algebraic form, if $a + b = b + a = c$

then $a = c - b$ and $b = c - a$

But here we note an essential limitation of the Natural Number system, namely, that such an expression as $a - b$ can only be used, *within the natural number system*, if $a > b$*; that is to say, only such natural number values of a and b give natural number values of the "difference" $a - b$.

(ii.) More generally, for Addition and Subtraction together, we have the algebraic form of expression—

$$a \pm b \pm c \pm d \pm \cdot \cdot \cdot \cdot \cdot \cdot$$

—subject, *within the natural number system*, to the conditions necessary in order that every subtraction, as it occurs, should give a natural number as its "result"; for example—

$$7 - 3 - 2 + 1 + 4 - 6 \cdot \cdot \cdot \cdot \cdot \cdot$$

And for this type of expression, we have "Laws of Commutation and Association in Addition and Subtraction," as follows:—†

(1) An equivalent expression may be obtained by altering the order of the steps—provided

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* The sign $>$, meaning *is greater than*, and the converse sign $<$, are used to express "order" of the numbers of the system; for example, $7 > 3$ and $6 < 9$.

† These "laws" are theorems, of which proofs are given in Appendix I.

(*a*) that only an *additive* term of the original expression may be given the first place in the new expression ;

(*b*) that all the new subtractions introduced by the change give natural numbers as their results.

(2) An equivalent expression may also be obtained by grouping the terms into sub-expressions of the same kind, under simple "laws of signs"—if, again, every new subtraction introduced by the change gives a natural number "result."

For example—

$$a - b + c - d - e + f - g = c - g - d + a + f - e - b$$

and $$= a - (b - c) - (d + e) + (f - g);$$

the standard simplest form for "evaluation" being—

$$(a + c + f) - (b + d + e + g);$$

for instance—

$$7 - 3 - 2 + 1 + 4 - 6 = 7 + 1 + 4 - 3 - 2 - 6$$
$$= (7 - 3) - (2 - 1) + 4 - 6 = 1.$$

4. **Multiplication** is, for the Natural Numbers, the special addition represented by—

$$a + a + a + \cdot \cdot \cdot \cdot \cdot \cdot$$

It is denoted by $a \times b$ or $b.a$,* if b denote the number of equal "terms." For example, $5 \times 3 = 3.5 = 5 + 5 + 5$.

It is easy to see (or to "prove") that—

$a \times b = b \times a$; for example, $5 \times 3 = 15 = 3 \times 5$; and that the extended multiplication $a \times b \times c \times d \times \cdot \cdot \cdot \cdot \cdot \cdot$ is subject to Laws of Commutation and Association, exactly like the Addition laws† ; for example [compare § 2]—

(1) $3 \times 5 \times 9 = 3 \times 9 \times 5 = 5 \times 3 \times 9 =$ etc.

(2) $3 \times 5 \times 2 \times 9 \times 6 = (3 \times 5) \times (2 \times 9) \times 6$
$= 3 \times (5 \times 2 \times 9) \times 6 =$ etc.,

giving a definite "product" of a given set of natural numbers.

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* See Appendix II. on the Multiplication sign.

† See Appendix I., for proofs.

5. (i.) **Division** is—and remains—the operation "inverse" to Multiplication; so that, if $a \times b = b \times a = c$,

we write also $a = c \div b$ and $b = c \div a$*;

for example, $3 = 15 \div 5$ and $5 = 15 \div 3$.

Like Subtraction, the previous inverse operation, it is, *within the Natural Number system*, a restricted operation—only, very much more restricted.

(ii.) Subject to these restrictions—namely, that every division, as it occurs, has a natural number as its "result"—there are "Laws of Commutation and Association in Multiplication and Division," exactly like those for Addition and Subtraction. For example [compare § 3]—

$$a \div b \times c \div d \div e \times f \div g = c \div g \div d \times a \times f \div e \div b$$
$$= a \div (b \div c) \div (d \times e) \times (f \div g)$$
$$= (a \times c \times f) \div (b \times d \times e \times g)$$

(See Appendix I.); for instance—

$$15 \div 3 \times 12 \div 2 \div 10 = 15 \times 12 \div 3 \div 2 \div 10$$
$$= (15 \div 3) \times 12 \div (2 \times 10) = 3.$$

6. (i.) The question whether Laws of the same kind hold for expressions involving all four operations—Addition, Subtraction, Multiplication and Division—is clearly a subject for consideration. For example, are the expressions—

$$a + b \times c - d \div e, \quad b \div e \times c + a - d, \quad a + b \times (c - d) \div e,$$

equivalent expressions, or are they not?

The answer is, of course, that they are not; and the first must therefore be *defined*—to mean

$$a + (b \times c) - (d \div e)$$

—the multiplications and divisions in such expressions being performed before the additions and subtractions, if not otherwise specifically required. If, for example, the opera-

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* Also written $a = \frac{c}{b}$ and $a = c/b$ —this last being, on the whole, the most serviceable; but all three notations have their uses.

tions were to be performed in the order in which they come, we should write—

$$^{*}(\,(a+b)\times c-d)\div e$$

—an expression which requires, for its "simplification," further "laws" which we shall now state :—

(ii.) For multiplication and division of an addition-and-subtraction expression we have "Laws of Distribution," which are easy to prove for the Natural Numbers, as follows :—

(1) $(a\pm b\pm c\pm\cdots)\times k=a\times k\pm b\times k\pm c+k\pm\ldots\ldots$ for example—

$$(7-3-2+1+4-6)\times 5=35-15-10+5+20-30$$

and, conversely—

$$(a\pm b\pm c\pm\cdots)\div k=a\div k\pm b\div k\pm c\div k\pm\cdots\cdots;$$

(2) For the case of multiplication this can be immediately extended to the multiplication together of any number of expressions of the form $a\pm b\pm c\pm d\pm\cdots$ the terms of the "distributed" expression being all the products of terms, one from each of the factor-expressions, and being additive or subtractive according as an even or an odd number of their factors come from subtractive terms of the factor-expressions.

7. These facts about Addition, Subtraction, Multiplication and Division—with their Laws of Commutation, Association and Distribution—are comparatively easy to prove (or, at any rate, to demonstrate convincingly) *if kept strictly within the Natural Number system.* (See Appendix 1.)

The first business of Elementary Arithmetic is with these operations and laws as actually applied to natural numbers in general. [See § 10, below]. And the first business of Elementary Algebra is familiarity with the laws in their general algebraic forms.

But before proceeding further in these directions, it is important to complete the definitions of the system of operations which have their origin in the properties of the Natural Numbers.

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* Different kinds of brackets for expressions of this kind are not necessary, nor even in the long run desirable ; but they are useful in elementary teaching.

8. (i.) **Involution, for the Natural Numbers, is an operation** which has the same relation to Multiplication that Multiplication itself has to Addition. It is the special multiplication represented by $a \times a \times a \times \cdots\cdots$; and it is denoted by a^b, in which b denotes the number of equal "factors." For example— $2^3 = 2 \times 2 \times 2 = 8$.

The quantity a is called the "base"; b the "exponent"; and a^b is called a "power of a."

(ii.) But the analogy with Multiplication (and with Addition) fails in this respect, that a^b is *not* equal to b^a; for example— $2^3 = 8$ and $3^2 = 9$.

This fact makes unimportant any extension of involution expressions, to more than two numbers. There is nothing here corresponding to the unique sum, or the unique product, of a number of given numbers. And, for the same reason, there is no important operation with the same relation to Involution that Involution has to Multiplication and Multiplication to Addition.

(iii.) It is easy to see that, for natural number values of a, b, c (and $b - c$, $a \div b$, where they occur),

$$a^{b+c} = a^b \times a^c \quad \text{and} \quad a^{b-c} = a^b \div a^c; \quad (a^b)^c = a^{b \times c} = (a^c)^b;$$
$$(a \times b)^c = a^c \times b^c \quad \text{and} \quad (a \div b)^c = a^c \div b^c.$$

For example (see §§ 4 and 5).

$$2^3 \times 2^5 = (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2 \times 2)$$
$$= 2 \times 2 \times \cdots \times 2 = 2^8$$
$$2^5 \div 2^3 = (2 \times 2 \times 2 \times 2 \times 2) \div (2 \times 2 \times 2) = 2 \times 2 = 2^2.$$
$$(2^3)^5 = (2 \times 2 \times 2) \times (2 \times 2 \times 2) \times (\cdots) \times (\cdots) \times (\cdots)$$
$$= 2 \times 2 \times 2 \times \cdots\cdots \times 2 = 2^{15} = (2^5)^3;$$
$$(3 \times 5)^2 = (3 \times 5) \times (3 \times 5) = (3 \times 3) \times (5 \times 5) = 3^2 \times 5^2$$
$$(9 \div 3)^2 = (9 \div 3) \times (9 \div 3) = (9 \times 9) \div (3 \times 3) = 9^2 \div 3^2$$

9. The fact that a^b is not equal to b^a has this further effect; that there are *two different operations "inverse" to Involution.*

If $a^b = c$, we write $a = \sqrt[b]{c}$ and $b = \log_a c$; for example—

$$2^3 = 8, \qquad 2 = \sqrt[3]{8}, \qquad 3 = \log_2 8.$$

The first of these two inverse operations is called **Evolution**; the other (too long denied a place among the operations of

Algebra) may be called "**Logarithmation**"—for want of a better name.*

The restrictions upon both of these operations—within the Natural Number system—are very great. But, in so far as they can be used, the following are practically obvious inverse forms of the Involution theorems of § 8, (iii.) :—

(1) $\sqrt[a]{(b \times c)} = \sqrt[a]{b} \times \sqrt[a]{c}$ and $\sqrt[a]{(b \div c)} = \sqrt[a]{b} \div \sqrt[a]{c}$;

$$a^{b \div c} = \sqrt[c]{a^b} \text{ and } \sqrt[a]{\sqrt[b]{c}} = \sqrt[p]{c} = \sqrt[b]{\sqrt[a]{c}}, \text{ if } p = a \times b.$$

For example, $\sqrt[2]{144} = \sqrt[2]{9} \times \sqrt[2]{16}$ and $\sqrt[2]{4} = \sqrt[2]{36} \div \sqrt[2]{9}$;

$$2^3 = \sqrt[4]{2^{12}} \text{ and } \sqrt[2]{\sqrt[3]{15625}} = \sqrt[6]{15625} = \sqrt[3]{\sqrt[2]{15625}}$$
$$(\sqrt[2]{25} = 5 = \sqrt[3]{125}).$$

(2) $$\log_a(b \times c) = \log_a b + \log_a c$$

and $$\log_a(b \div c) = \log_a b - \log_a c;$$

$$\log_a b^c = c.\log_a b \text{ and } \log_a b \times \log_b c = \log_a c$$
$$\text{or } \log_b c = \log_a c \div \log_a b.$$

For example, $\log_2 256 = \log_2 8 + \log_2 32$

and $$\log_2 4 = \log_2 32 - \log_2 8;$$

$$\log_2 4096 = 3.\log_2 16 = \log_2 16 \times \log_{16} 4096$$

and $$\log_{16} 4096 = \log_2 4096 \div \log_2 16.$$

[*Note.*—The obvious very careful adjustment of the numbers in these examples indicates clearly the nature of the restrictions on *elementary* use of these two inverse operations. The extent to which the numbers 2 and 3 are used in them—as evolution-indices or bases—is significant. Examples become rapidly more cumbrous as base and exponent (and index) numbers are increased.]

10. (i.) The **Decimal** (or positional) **Notation** for the Natural Numbers is a powerful mathematical instrument. Its uses in the operations of Addition, Subtraction and Multiplication are based upon the Laws of §§ 2–6 above.†

* The term "Logarithmation" is not in general use. But it will be found, for instance, in Steinmetz' "Engineering Mathematics" (3rd Edn.), Ch. I., § 15, p. 20.

† Knowledge of the *use* of this notation is, of course, assumed in foregoing sections, *i.e.*, knowledge of the elementary practice of Arithmetic by mechanical rules. We are here concerned with the arithmetical and algebraical principles underlying these elementary rules of Arithmetic.

We use a small number of special symbols—the "figures" 1, 2, 3, ... 9 and 0. The natural numbers *one*, *two*, *three*, ... *nine* are denoted by 1, 2, 3 . . . . 9. Beyond these the principle of "position" is used; and in the application of that principle the symbol 0 plays an important part—of which the significance will appear more clearly as we proceed. The number *ten* is denoted by the double-symbol 10; *eleven* by 11, which means $10 + 1$; etc.; *nineteen* by 19, which means $10 + 9$; then *twenty* by 20, which means 2.10; *twenty-one* by 21, which means $2.10 + 1$; etc., for example, *seventy-four*, which means $7.10 + 4$; *ninety-nine* by 99, which means $9.10 + 9$; then, again, *one hundred* by 100, which means 10.10 or 10^2; *one hundred and one* by 101, which means $100 + 1$ or $10^2 + 1$; and so on; for example, *three hundred and fifty-six* by 356, which means $3.10^2 + 5.10 + 6$; and *four thousand, eight hundred and sixty-five* by 4865, which means $4.10^3 + 8.10^2 + 6.10 + 5$.

The constituents of the composite decimal symbol for a number are called its "digits"—being units digit, tens digit, hundreds digit, etc.; for example, 6, 5, 3 respectively for the number denoted by 356; 1, 2, 0, 5 for the number denoted by 5021.

And we note that, for the most part, the names—such as "seventy-four" and "three hundred and fifty-six"—are just an expression *in words* of what the notation more scientifically expresses. The resources of ordinary language for this purpose are soon exhausted—even in the Hindu language, which is specially (and, of course, quite unnecessarily) rich in number words; but the decimal notation is inexhaustible: it is adequate to the essential "infinity" of the natural number system.

(ii.) (1) The special significance of the symbol 0 is to be emphasised. Its relatively recent introduction is one of the great landmarks of mathematical science. The peculiar importance of its use is that it serves to maintain the essential principle of position when none of the original number symbols (1, 2, · · · 9) can be used for that purpose; for example, in 5021, which, expressed verbally, reads "five thousand and twenty-one"—with a gap between the "tens" and the "thousands." The distinction between 5021 and

521 (and, more generally, 500 · · 200 · · 100 · ·) is a perfect economy of notation.

(2) But this fact clearly carries with it a "number" meaning for the symbol 0, *when used alone*. For the notation 5021, for example, meaning $5.10^3 + 0.10^2 + 2.10 + 1$—following the general principle stated in (i.)—amplifies the verbal expression "five thousand and twenty-one" by the equivalent of some such phrase as "no hundreds," or "*nought* hundred." And thus we come at the "number" *nought* (denoted by 0), which is such that

$0 + 0 + 0 + \cdot \cdot \cdot = n.0 = 0 = 0.n, \quad n + 0 = n = 0 + n,$* etc.,

for all natural number values of n. Its relation, in addition and subtraction, to 1 gives it—at any rate, in a certain mathematical sense—a place *at the beginning of the Natural Number sequence*, so that we write 0, 1, 2, 3 · · · · · as amplified form of that sequence.

(3) The use of the symbol 0 is, of course, independent of the number of the other special number symbols, 1, 2, etc. If we use eleven others, instead of nine, we have "the duodecimal system"—in which it is the number *we* call "twelve" which is denoted by 10. If we used only one other, viz., 1, the natural numbers one, two, three, four, five, six . . . . would be denoted by 1, 10, 11, 100, 101, 110, . . .

It is perhaps worth noting that the number of digits in the decimal symbol for any given number may be increased to any extent we please, by *prefixing* "noughts"; for example—

$$5021 = 0005021.$$

(iii.) The addition† of two natural numbers, so denoted—for example, 327 and 4865—is a simple application of the "Laws of Algebra," as follows :—

$$\left.\begin{matrix} 4865 \\ +327 \end{matrix}\right\} = \left\{\begin{matrix} 4.10^3 + 8.10^2 + 6.10 + 5 \\ \qquad + 3.10^2 + 2.10 + 7 \end{matrix}\right.$$
$$= 4.10^3 + 11.10^2 + 8.10 + 12.$$
$$= 4.10^3 + (10 + 1).10^2 + 8.10 + (10 + 2)$$
$$= 5.10^3 + 10^2 + 9.10 + 2.$$
$$= 5192,$$

in which the basis of the ordinary arithmetical rule is apparent.

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* See Appendix III. on 0 and ∞.

† The elementary "Addition Table" and "Multiplication Table"—for addition and multiplication of the basic numbers 1, 2, . . . 9—are, of course, assumed as a basis.

(iv.) So, again, for subtraction; for example,

$$\left.\begin{array}{r}4527\\-865\end{array}\right\}=\left\{\begin{array}{r}4.10^3+5.10^2+2.10+7.\\-8.10^2-6.10-5.\end{array}\right.$$

(reversing) $=(7-5)+(10+2-6).10+(10+4-8).10^2+3.10^3$

(re-reversing) $= 3.10^3+6.10^2+6.10+2.$

$= 3662,$

in which, again, the basis of the arithmetical rule is apparent.

(v.) For multiplication, we proceed in two stages :—

(1) for example,

$$327 \times 9 = 9.(3.10^2+2.10+7)$$
$$= 27.10^2+18.10+63,$$
by Law of Distribution [§ 6, (ii.)]
$$= 27.10^2+(18+6).10+3.$$
$$= 27.10^2+(20+4).10+3.$$
$$= 29.10^2+4.10+3.$$
$$= 2.10^3+9.10^2+4.10+3$$
$$= 2943,$$

in which, once again, the basis of a familiar arithmetical rule is apparent;

(2) then, again, for example—

$$4865 \times 327 = 4865 \times (3.10^2+2.10+7).$$
$$= 4865 \times 3 \times 10^2 + 4865 \times 2 \times 10 + 4865 \times 7,$$

and this gives, using the rule of (1)—

```
   1 4 5 9 5[0 0]
+      9 7 3 0[0]
+      3 4 0 5 5
 ----------------
=1 5 9 0 8 5 5,
```

again making apparent the basis of the arithmetical rule, in which the terms of the final step are commonly taken in *either* of the two opposite orders (and the "noughts" in brackets omitted).

11. (i) The "Division-transformation" is a process of successive subtractions, "inverse" to the process of additions by which multiplication of natural numbers (see § 10, (v.)) is performed.

Given natural number values of a and b, such that $a > b$, the problem is to find the greatest multiple of b which is not greater than a;
or, algebraically, to find natural number values of c and d such that—

$$a = b \times c + d \quad \text{and} \quad d < b.$$

Using the decimal notation, the successive digits for the number c are found as in the following example:—

If $a = 84613$ and $b = 327$, we have

$$84613 = 327 \times 2 \times 10^2 + 19213$$

the primary fact at this stage being that—

$$b \times 2 \times 10^2 < a < b \times 3 \times 10^2.$$

$$19213 = 327 \times 5 \times 10 + 2863$$

$$2863 = 327 \times 8 + 247$$

Hence $84613 = 327 \times 258 + 247$; *i.e.*, $c = 258$ and $d = 247$.

[*Note.*—The first step, in this example, finds the "hundreds" digit for c, the second the "tens" digit, the third the "units" digit. These three digits then determine the required multiplier of b. The details of the process are set out—in "long division"—so as to avoid repetition of the "remainder" numbers 19213, etc.].

(ii.) Since, in (i.), a common "factor" of a and b must also be a factor of d, the "Highest Common Factor" of a and b is obtainable by a systematic succession of division-transformations.

(iii.) Of less practical importance, but of considerable interest as an application of similar principles, is the arithmetical rule for finding "Square Roots."

With a view to later developments, the problem may be stated in the following way—

Given a natural number, N, find another, S, such that

$$S^2 < N < (S + 1)^2.$$

Using, for S, the general polynomial form—

$$a.t^n + b.t^{n-1} + c.t^{n-2} + \cdot\cdot\cdot + k.t + l$$
$$(= A + B + C + \cdot\cdot + K + l).$$

in which $t = 10$ and $a, b, c \ldots . k, l$ are natural numbers, each less than 10—we first find n and a, "by inspection," so that

$$(a.t^n)^2 < N < ((a+1).t^n)^2 \quad \text{or} \quad A^2 < N < A'^2.$$

We then to proceed to find b, so that—

$$(a.t^n + b.t^{n-1})^2 < N < (a.t^n + (b+1).t^{n-1})^2$$

or $$(A+B)^2 < N < (A+B')^2$$

whence $$(2.A+B).B < N - A^2 < (2.A+B').B'$$

say $$T_1 < N_1 < T_1'$$

—a relation from which b (or B) can be determined "by trial." (See the example, below.)

So, again, c from

$$(2.A + 2.B + C).C < N_2 < (2.A + 2.B + C').C'$$

or $$T_2 < N_2 < T_2'$$

where $$N_2 = N - (A+B)^2 = N_1 - T_1,$$

etc.

For example, if $N = 74136529$, then $n = 3$ and $a = 8$;

$$N_1 = N - A^2 = N - 64 \times 10^6 = 10136529,$$

and "trial" gives $b = 6$, whence

$$(2.A+B).B = (166 \times 10^2) \times (6 \times 10^2) = 996 \times 10^4 = T_1$$

Hence $N_2 = N_1 - T_1 = 176529$, and trial gives $c = 1$,

whence $(2.A + 2.B + C).C = (1721 \times 10) \times (1 \times 10)$
$$= 1721 \times 10^2 = T_2$$

Finally $N_3 = N_2 - T_2 = 4429$, and $d = 0$.

This gives the result, namely, $8610^2 < N < 8611^2$.

The arithmetical rule gives an obvious simple setting out of the numerical details.

CHAPTER II.

THE GENERAL NUMBERS OF MATHEMATICS.

12. (i.) The general mathematical system of numbers is built up on the foundation of number ideas derived from the study of the Natural Numbers. It fulfils two important requirements, namely :—

(1) it provides fully for *measurement* of quantities of a given kind, in terms of a unit quantity of that kind (for example, lengths or angles) ;

(2) the new types of "number," and the extensions of the "operations" which they require, are such that *no restrictions* are necessary in the application of the "operations" to the "numbers."

But while the whole fundamental theory of the Natural Numbers is (as we have seen) quite simple and elementary, the general mathematical theory is long and involved, and in certain respects very difficult. This is why *it is important—in order to ensure clear ideas about Number in general—to keep the theory of the Natural Numbers quite distinct from the more general theory.* It is comparatively easy to know everything of fundamental importance about the Natural Numbers, *if they are kept strictly to themselves* ; it is not at all easy to know everything of fundamental importance about the general numbers of Mathematics. But, as all number ideas can ultimately be traced back to natural number ideas, the difficulties of the general theory may be minimised by obtaining a thorough knowledge of the elementary theory—in the way indicated in foregoing sections.

(ii.) The mathematical system consists of :

(1) the "Integral Numbers," positive and negative ;

(2) the "Fractional Numbers," positive and negative—Integral and Fractional Numbers together forming the system of "Rational Numbers" ;

(3) the "Irrational Numbers," positive and negative—Rational and Irrational Numbers together forming the system of "Real Numbers";

(4) the "Unreal Numbers"—Real and Unreal Numbers together forming the system of "Complex Numbers," which is the general mathematical system of numbers. (The "Imaginary Numbers" are a particular class of Unreal Numbers, formed as products of the real numbers with the number ι such that $\iota^2 = -1$; see Ch. VI., § 25 (iii.)).*

[*Note.*—All the names used, except "Integral" and "Fractional," have been unfortunately chosen. "Rational" and "Irrational" were probably never meant to have any connection with the everyday use of these terms; the root-word is *ratio*, but the relationship with it is mathematically unsound.† "Real," "Unreal," and "Imaginary" convey a quite exaggerated antithesis—based on what is little more than different degrees of familiarity with equally "real" uses of these several types of Number. The student should accustom himself as quickly as possible to use them merely as defined technicalities of mathematical science.]

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* We shall find that it is best to define, in this connection, first the "Imaginary Numbers," then the general "Complex Numbers." The "Unreal Numbers" are best thought of as the complex numbers which are not "real."

† See Ch. V., § 20.

CHAPTER III.

THE INTEGRAL NUMBERS.

13. The Positive Integral Numbers are, in ordinary usage, not distinguished from the Natural Numbers; but it is much better that they should be, for reasons indicated in § 12 (i.). They belong essentially to the general mathematical system outlined in § 12 (ii.).

The proper distinction consists in the fact that while the properties of the Positive Integral Numbers are primarily defined directly from those of the Natural Numbers, these new numbers are gradually and progressively—in the development of the mathematical system—freed from the restrictions which are to be regarded as characteristic of the Natural Numbers. The distinction is not, of course, a *practical* necessity; but its *theoretical* importance can hardly be exaggerated—towards a sound understanding of mathematical principles.

14. (i.) The Positive Integral Numbers and the Negative Integral Numbers—say *positive* 1, *positive* 2, ... and *negative* 1, *negative* 2, ... —have (*each* sub-system) properties in Addition and Subtraction derived directly from the Natural Numbers; for example—

$$pos\ 3 + pos\ 5 = pos\ 5 + pos\ 3 = pos\ 8,$$
$$neg\ 3 + neg\ 5 = neg\ 5 + neg\ 3 = neg\ 8;$$
$$pos\ 5 - pos\ 3 = pos\ 2;\ \ neg\ 5 - neg\ 3 = neg\ 2.*$$

* The notation " + 3," " − 5" is used for " *positive* 3," " *negative* 5," and *should be so read*; for example, " $(+3) - (-5)$ " should be read " *positive* 3 minus *negative* 5—implying two quite distinct, though of course closely related, uses of " + " and " − ". There is, further, a third quite distinct use of the sign " −," namely, meaning " *opposite*"; for example, + 3 and − 3 are *opposite* numbers, and we may write $-3 = -(+3)$ or $+3 = -(-3)$—a form which is most familiar in the equivalence of the algebraic relations $x + y = 0$ and $y = -x$ (meaning that y is *opposite* in value to x).

Note that the phrase " equal *and* opposite" is a contradiction in terms, besides being uneconomical in its use of words; " opposite in sign" *is* a useful phrase for the qualitative relationship it expresses.

The +, meaning *positive*, is commonly omitted. This emphasises the specially intimate relation between the Positive Integers and the Natural Numbers.

But they are combined into one system—the system of Integral Numbers—by means of the "number" *nought*, related to them according to the following propositions [See § 10, (ii.)]:—

$$0 = \text{pos } a - \text{pos } a = \text{neg } a - \text{neg } a$$
$$= \text{pos } a + \text{neg } a = \text{neg } a + \text{pos } a$$

of which the inverse forms are —

$$\text{pos } a = 0 + \text{pos } a = \text{pos } a \pm 0 = 0 - \text{neg } a$$
$$\text{neg } a = 0 + \text{neg } a = \text{neg } a \pm 0 = 0 - \text{pos } a$$

And the operations of Addition and Subtraction (so amplified), with the Laws of Commutation and Association which are regarded as characteristic of these operations, may then be applied *without restriction* to the Integral Numbers. Thus

$$(+3)-(-5)=(+3)-(0-(+5))=(+3)-0+(+5)$$
$$=(+3)+(+5)=+8; \quad \text{and} \quad (-3)-(+5)=-8;$$
$$(+3)-(+5)=(+3)-((+3)+(+2))=(+3)-(+3)-(+2)$$
$$= \quad 0-(+2)= \; -2; \quad \text{and} \quad (-3)-(-5)=+2;$$
$$(+3)+(-5)=(+3)+((-3)+(-2))=(+3)+(-3)+(-2)$$
$$= \quad 0+(-2)=-2; \quad \text{and} \quad (+5)+(-3)=+2.$$

A practical principle of very great importance is that *addition of any integral number is equivalent to subtraction of the opposite integral number*, and *vice versa.*

(ii.) So far as Addition and Subtraction are concerned, the Negative Integers are distinguished from the Positive Integers only relatively, not absolutely: these two sub-systems have, with respect to these operations, both the same relation to the Natural Number system. But for Multiplication it is not found possible to maintain a similar position. This operation is first defined for *one* of the two sub-systems, by analogy with multiplication of natural numbers; and the proper completion of the definition is then *deduced.* Choice of the Positive Integers for this purpose is the beginning of a process of establishing a special relationship* between the Natural Numbers and the Positive Integers.

By hypothesis, (1) multiplication of a positive integer by a positive integer is simply a particular addition; hence (2) multiplication of a negative integer *by* a positive integer may also be *defined* as a particular addition—for example, $(-5) \times (+3) = (-5) + (-5) + (-5) = -15$; (3) regarding the Law of Commutation as characteristic of Multiplication, we therefore *define*, for example, $(+3) \times (-5)$ to be equal to $(-5) \times (+3)$, that is, (-15)—from which we have the principle that multiplication *by* a negative integer has the opposite effect (gives the opposite result) to multiplication by the opposite positive integer; hence (4) the *definition* of Multiplication is completed by, for example, $(-5) \times (-3) = -\{(-5) \times (+3)\} = +15$.†

Thus Multiplication begins to have a meaning to some extent independent of its original purely addition meaning; and this independence increases at later stages of the theory.

(iii.) We have now operations of Addition, Subtraction and Multiplication for the Integral Numbers, which are quite unrestricted in their application to these numbers—all such operations on integral numbers producing integral results.

These three operations are, therefore, called "**the Integral Operations.**"

The restrictions on Division of integral numbers (defined as the inverse of Multiplication) are practically the same as for natural numbers.

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* The relationship is so close that the Positive Integers are not commonly distinguished from the Natural Numbers. (See § 13).

† The definition requires supplementing by, for example,

$$0 \times (\pm 3) = 0 = (\pm 3) \times 0; \text{ see § 10, (ii.).}$$

It may be here noted that the Negative *Integral* Numbers—*in themselves*—appear to have very little practical application (more particularly, their Multiplication properties). Their practical importance consists in the fact that they are part of the general Number-system—in which the Negative Numbers are as indispensable as the Positive Numbers. See Note at the end of Ch. IV. (p. 28).

(iv.) Involution, being for the Natural Numbers simply a special multiplication, is defined *for positive integral exponents* in the same way—the *base* being either positive or negative.

We have then [as in § 8, (iii.)] :—

$$a^{b+c} = a^b \times a^c, \quad a^{b \times c} = (a^b)^c = (a^c)^b, \quad (a \times b)^c = a^c \times b^c$$

for integral values of the base quantities and *positive* integral values of the exponent quantities. And these give the following inverse forms—restricted, as shown, at this stage :—

$$a^{b-c} = a^b \div a^c, \text{ if } b > c;$$

$$a^{b \div c} = \sqrt[c]{a^b}, \text{ if } b \text{ divisible by } c;$$

$$(a \div b)^c = a^c \div b^c, \text{ if } a \text{ divisible by } b.$$

The extension to negative integral exponents cannot be made until the Fractional Numbers have been defined and brought into use. But we may note here that $a^{b-c} = a^b/a^c$ is used, in the extreme case when $b = c$, to give the *definition* $a^0 = +1$, *for all the values of* a.*

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* A definition which may clearly also be applied to the Natural Number case.

CHAPTER IV.

THE RATIONAL NUMBERS.

15. (i.) The extension of the mathematical number-system from Integral to "Rational"—and, subsequently, to "Real"—Numbers, comes, as a matter of practical experience, from the application of numbers to the "measurement" of Physical Quantities.

In the most elementary measurement—of a length, or an angle—we use first some chosen "unit" length, or angle, and find the nearest "integral multiple," of that unit, less than the quantity being measured. We then use a sub-unit, which is some "sub-multiple" of the original unit, and apply it in the same way to the part which was left "unmeasured" at the first stage; and so on. For example, we may use, for lengths, *yard*, *foot*, *inch*, *eighth of inch*; *metre*, *decimetre*, *centimetre*, *millimetre*; and, for angles, *degree*, *minute*, *second*, *tenth of second*; etc. Thus: angle $AOB = 35° 42' 31''$.

But it is a very important scientific advance to realise that the same facts of measurement may be expressed by the use of only *one* unit quantity (of the kind in question) and *one* "rational" number—which may be either integral or "fractional." This number, when its existence has been established, is called the "ratio"* *of* the quantity measured to the unit quantity—or, again, the "measure" of the original quantity in terms of the unit.

(ii.) But in order to establish these principles of Measurement, the Fractional Numbers have to be defined in some purely number way, and their existence, *as "numbers,"* then further justified by showing that they can be used—along with the Integral Numbers—in properly defined operations of Addition, Subtraction, Multiplication, etc.

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* For an elementary discussion of Ratio, see "A First Trigonometry," by Waddell & Picken (Melville & Mullen, Melbourne), Ch. I. For a fuller treatment, see article on "Ratio and Proportion," by the writer, in the Mathematical Gazette, January and May, 1920

(iii.) *A fractional number is defined as the quotient of two integral numbers—when such quotient is not itself integral.* For example, a "number," f, is defined such that

$$f = 5 \div 3, \text{ and, inversely, } f \times 3 = 5$$

the definition of this number thus including definition of *its* multiplication *by* 3.

Regarding Association and Commutation as essentially characteristic of Multiplication, we shall amplify the definition by, for example,

$$(f \times 3) \times 28 = f \times (3 \times 28) = f \times 84\,;$$

whence $$f \times 84 = 5 \times 28 = 140, \text{ and } f = 140 \div 84.$$

Thus every rational number (whether integral or fractional) has an infinity of different (but, of course, equivalent) "fractional expressions"—obtainable by multiplying (or dividing) *both* numerator and denominator, of any one of its fractional expressions, by the same integer. (The division of numerator and denominator by a common factor is specially familiar, in the elementary process of reducing "fractions" to simpler forms, by "cancelling" common factors of numerator and denominator).

The simplest—or irreducible—fractional form (when the number so expressed is not integral) is used to *denote* the fractional number; for example, $f = 5/3$. The "mixed" form $1\,{}^2/_3$, used to some extent in Arithmetic—meaning $1 + {}^2/_3$*—is essentially a secondary form (of no particular mathematical importance).

The example used above is one in which the two integers involved are both positive; and the fractional number is defined in that case to be also *positive*—in accordance with the principles of "sign" established in § 14 (ii.). When one of the integers is positive and the other negative, the fractional number is defined to be *negative*†; when both integers are negative, it is again *positive*.

Two rational numbers such as $+5/3$ and $+3/5$ (or $-5/3$ and $-3/5$) are said to be "reciprocal."

(iv.) From the definition of the Fractional Numbers, and the

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* See Appendix II, (ii), (2).

† See Note on the practical application of the Negative Numbers at the end of this Chapter.

principles of Association and Commutation, we have the proper *definitions* of Multiplication and Division—for Rational Numbers (Integral or Fractional).* For example,†

$$5/3 \times 7/13 = (5 \div 3) \times (7 \div 13) = (5 \times 7) \div (3 \times 13) = 35/39$$

and

$$5/3 \div 7/13 = (5 \div 3) \div (7 \div 13) = (5 \times 13) \div (3 \times 7) = 65/21$$

—types of processes in which "cancelling" is commonly done: equal factors in numerator and denominator expressions being cancelled, because together equivalent, when "associated" together, to a "factor" 1 of the expression.

The product of two "reciprocal" rational numbers $= +1$; and multiplication by either is equivalent to division by the other.

(v.) To define Addition and Subtraction of Rational Numbers, the Law of Distribution in Division is taken as characteristic—giving :—

(1) for example,

$$5/3 + 13/3 = 18/3 = 6 \text{ and } 5/3 - 13/3 = -8/3$$

when the fractions have the same denominator; and

(2)—reducing the general case to case (1), by means of (iii.) above—for example,

$$5/3 + 7/13 = 65/39 + 21/39 = 86/39,$$
$$5/3 - 11/15 = 25/15 - 11/15 = 14/15,$$

—the "least common multiple" (L.C.M.) of the original denominators being used as "common denominator" for the equivalent fractional forms introduced.

(vi.) The four operations Addition, Subtraction, Multiplication and Division—with their characteristic Laws of Commutation, Association and Distribution—are then applicable, *without restriction*, to the Rational Numbers, so as to produce "rational" results. For this reason, these four operations are called **"the Rational Operations."**

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* The special relation of the Fractional Numbers to the operation of Division makes it simpler to take Multiplication and Division of the Rational Numbers before Addition and Subtraction. See, again, Ch. VI., § 25 (iii.).

† For the basic form see Ch. I., § 5, (ii.). Questions of "sign" of the numbers, not illustrated in sub-sections (iv.) and (v.), are dealt with on the principles established in § 14, (i.) and (ii.)

(vii.) **The Rational Numbers have a new Number property** of great mathematical importance—as follows :—

Between any two rational numbers there is an infinity of other rational numbers.

This is an immediate consequence of the principle of the "common denominator," and of the fact that the common denominator may be magnified to any extent we please. For example, to take an important special case :—

Between 0 and +1 there is no integral number; but, expressing fractionally,

the denominator 2 gives $0 < \frac{1}{2} < 1$

" " 3 " $0 < \frac{1}{3} < \frac{2}{3} < 1$

. . . .

" " 1000000 gives

$0 < 1/1000000 < 2/1000000 < \cdots\cdots < 999999/1000000 < 1,$

etc.

We note that if we take this process as far as we please in any case, and then take two of the intermediate numbers which *seem* so much closer than the original ones—for example, if we take 75643/1000000 and 75644/1000000—there is still an infinity of rational numbers between these.

A system of numbers which has this remarkable property is said to be "dense."*

Referring back to the example just used, we observe that there is no limit to the nearness to 0, of rational numbers—or, to "smallness" of such numbers. And the general fact of "denseness" may clearly be expressed in terms of "smallness" of *differences*. This is a principle of great importance in Higher Mathematics.

(viii.) It is important to have in mind a scheme of all the Rational Numbers, in tabular form—as follows :—

$\cdots -2, \quad -1, \quad 0, \quad +1, \quad +2, \ldots$

$\cdots -2, \ -\frac{3}{2}, \ -1, \ -\frac{1}{2}, \ 0, \ +\frac{1}{2}, \ +1, \ +\frac{3}{2} \ +2, \ldots$

$\cdots -2, -\frac{5}{3}, -\frac{4}{3}, -1, -\frac{2}{3}, -\frac{1}{3}, 0, +\frac{1}{3}, +\frac{2}{3}, +1, +\frac{4}{3}, +\frac{5}{3}, +2, \ldots$

etc.

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* This is merely a definition of the use of the term "dense" in relation to the Rational Numbers. The precise definition of "denseness," in general, is not within the scope of this book.

The first line consists of the Integral Numbers; the second, of those rational numbers which can be expressed fractionally with denominator 2; the third, similarly, with denominator 3; etc. Each line, after the first, includes numbers already tabulated; in particular, the Integral Numbers are common to all the lines; but each line also includes numbers not previously tabulated.

(Another interesting tabulation—of less fundamental importance—is as follows:—

$$\tfrac{1}{1};\ \tfrac{1}{2},\ \tfrac{2}{1};\ \tfrac{1}{3},\ \tfrac{2}{2},\ \tfrac{3}{1};\ \tfrac{1}{4},\ \tfrac{2}{3},\ \tfrac{3}{2},\ \tfrac{4}{1};\ \text{etc.}$$

characterised by the fact that each group has a specific sum of numerator and denominator).

16. (i.) The **Decimal Numbers** are a very important subsystem of the Rational Numbers — consisting of (1) the Integral Numbers and (2) those fractional numbers which can be expressed with denominators which are powers of 10.

(ii.) These numbers have this very great convenience, that the decimal notation for the Natural Numbers can be very simply adapted to them, and the elementary rules of operations upon the Natural Numbers correspondingly adapted—as follows:—[See Ch. I, §10.

(1) $898/10^2$, for example, is written 8·98—the number of digits after "the decimal point" being equal to the exponent of the power of 10 which is the denominator of the fraction;

(2) $452{\cdot}7 - 8{\cdot}98\ [= 4527/10 - 898/10^2$
$= (45270 - 898)/10^2]^* = 443{\cdot}72$

is a typical example of Subtraction (and, so, Addition);

(3) $4{\cdot}865 \times 3{\cdot}27\ [= 4865/10^3 \times 327/10^2$
$= 4865 \times 327/10^5]^* = 15{\cdot}90855$

is a typical example of Multiplication.

(Division is not quite the same kind of question, because it is not an Integral Operation; but see § 17, (ii.) below).

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* The steps within the square brackets may clearly be omitted, and replaced by simple rules for the placing of the decimal point.

For Addition and Subtraction, we may "reduce" to decimal expressions with the same number of digits before, and the same number of digits after, the decimal point—by prefixing noughts to the digits *before* [see § 10, (ii.), (3)], and affixing noughts to the digits *after*, the decimal point (this latter, from the definition, clearly leaving the number unchanged). For example, $452{\cdot}7 \pm 8{\cdot}98 = 452{\cdot}70 \pm 008{\cdot}98$.

(iii.) It is a striking mathematical fact that this Decimal sub-system of the Rational Numbers is also a "dense" system. The argument of §15, (vii.) can be adapted in an obvious way to the Decimal Numbers.

17. (i.) *In actual practice nearly all numerical calculation is performed in terms of the Decimal Numbers.*

This is brought about by the principle of "Decimal Approximation"—a principle which is *convenient* for use in processes dealing with rational numbers, but *practically essential* when irrational numbers are involved. See Ch. V., § 20.

The principle is that a decimal number can always be found which differs by less than any number, however small,* we care to name, from a given real number; and that such "approximations" can be used, in operations, to give "approximate results"—the accuracy of which depends in a definite way on the degree of original approximation.

(ii.) The decimal approximations to a given fractional number may be systematically determined by means of "division-transformation". (See Ch. I., § 11).

For example,

$$\begin{array}{rlll} & 84613/327 & > 258 & \text{and } < 259 \\ = & (846130/327)/10 & > 2587/10 & \text{and } < 2588/10 \\ & \quad or & > 258{\cdot}7 & \text{and } < 258{\cdot}8 \\ = & (8461300/327)/10^2 & > 25875/10^2 & \text{and } < 25876/10^2 \\ & \quad or & > 258{\cdot}75 & \text{and } < 258{\cdot}76 \end{array}$$

and so on—the process having no end, if the original denominator has prime factors other than 2 and 5. (A simple standard way of setting out the division process is familiar).

But since there is only a limited number of possible "remainders"† for the several division-transformations of the process, one which has occurred must ultimately occur again in precisely similar circumstances—and all the steps thereafter will then be a recurrence, infinitely repeated. Thus *all fractional numbers which are not themselves decimal numbers, have "infinite recurring" decimal expressions.* For example,

$$47/6 = 7{\cdot}8\dot{3}\,; \quad 747/3500 = {\cdot}213\dot{4}2857\dot{1}\,;$$

* See § 15, (vii.).

† The positive integers less than the denominator.

(In the first of these, since $47/6 = 235/30 = (23{\cdot}5)/3$, the number of digits in the "recurring period" must be less than 3; it is actually 1. In the second, since

$$747/3500 = 1494/7000 = 1{\cdot}494/7,$$

the number must be less than 7, and is therefore as great as it could possibly be).

The process of this sub-section may clearly be adapted to the division of any one decimal number by any other. For example,

$$\cdot 00747 \div 3{\cdot}5 = 747/350000 = \cdot 00213\dot{4}2857\dot{1}.$$

(iii.) The converse proposition, that *every "recurring decimal" is the decimal expression of a fractional number*, follows at once from the fact that the recurring period yields an infinite Geometric series which has for its ratio of progression a power of $1/10$. This theoretical fact gives, of course, the familiar rule for the conversion from recurring decimal to common fraction.

(iv.) Decimal approximation is clearly sufficient for all purposes of practical measurement—because of (1) the "denseness" of the Decimal Number system, and (2) the essentially limited precision of practical measuring instruments.

18. The definition of Involution for exponents which are positive integral (or 0)—as stated in § 14, (iv.)—is equally applicable to bases which are fractional; for example,

$$(-2/3)^5 = -32/243$$

And it may now be extended to negative integral exponents.

For this purpose we use the identity $a^{b-c} = a^b \div a^c$, which holds if b and c are positive integral and $b > c$. If $b < c$, the expression $a^b \div a^c$ has *now* an equivalent $1 \div a^{c-b}$; for example, $2^{+4} \div 2^{+9} = 1 \div 2^{+5} = 1/32$. Hence a^{b-c}—so far meaningless in this case—is *defined* to be equal to $1 \div a^{c-b}$; or $a^{-n} = 1/a^{+n}$, if a represents all the rational numbers and $\pm n$ all the integral numbers; for example,

$$(-2/3)^{-5} = -243/32.$$

It is then easy to *prove* that the theorems

$$a^{b+c} = a^b \times a^c \text{ and } a^{b-c} = a^b \div a^c; \; a^{b\times c} = (a^b)^c = (a^c)^b;$$

$$(a \times b)^c = a^c \times b^c \text{ and } (a \div b)^c = a^c \div b^c$$

hold good for *rational* values of the *base*-quantities and *integral* values of the *exponent*-quantities, *without restriction.* But restrictions remain upon the use of $a^{b\div c}$,* because involution with fractional exponents has not yet been defined. The extension to this case essentially involves irrational numbers.

This section ends the discussion of the principles of "Elementary" Arithmetic and Algebra.

Note on Application of the Negative Numbers to Measurement—

Both Positive and Negative Numbers are required for the measurement of quantities which occur in two "opposite" kinds—for example, *credit* and *debit*, and *quantities of electricity*.

Their most fundamental and most important use is geometrical, in relation to the two "opposite *directions*" of a given straight line—or the two "opposite *senses* of circulation" in a given plane (which may be correlated with the two opposite *directions* "normal" to the plane). "Vector quantities," of a given kind, which are restricted in direction to two such directions, are specifiable by positive and negative numbers—as their measures, in terms of one of them (chosen as unit). The application of this principle, in Plane Geometry, (1) to angles, (2) to line-vectors (or "directed lengths"), is the basis of Elementary Trigonometry and of Plane Analytical Geometry—branches of mathematical science in which the Negative Numbers have equal importance with the Positive Numbers.

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* Restricted here to *integral* values (positive or negative) of $b \div c$, as well as of b and c.

PART II.

THE REAL NUMBERS
AND
THE COMPLEX NUMBERS.

CHAPTER V.

THE REAL NUMBERS.

19. The theory of the extension of the Number-system to the Irrational Numbers has essential difficulties. But a good working knowledge of these numbers, for practical purposes, is not beyond the scope of the average student of Mathematics; and such a knowledge is in fact necessary to any sound grasp of modern mathematical ideas.

Three things discussed in previous sections pave the way for this more difficult step :—

(1) The principle that numbers actually exist to correspond with the demands of the several inverse operations : the Integral Numbers satisfying the need of Subtraction ; the Rational Numbers, of Subtraction and Division. We are led to expect something of the same kind when we take account further of Evolution and Logarithmation.

(2) The principle of "Measurement," by numbers ; or the principle of Ratio.

(3) The principle of Decimal Approximation—to numbers which are not themselves decimal numbers.

The *theory* of the Integral, and of the Rational, Numbers is based directly on the first of these principles. Their *practical importance* is chiefly a consequence of the second.

But while "the Real Numbers" do in fact go a long way (if not all the way) to meet the need of the last two inverse operations, they are not so closely or directly related to them. They are defined in terms of the second principle—that of Ratio—by means of the third—that of Approximation, which has so far been expressed only *decimally*, but is now to be expressed *rationally*.

20. (i.) The Fractional Numbers, supplementing the Integral Numbers, were the first requirement of the principle of Measurement. But it is easy to see that—theoretically,

at any rate—they are not altogether sufficient to meet that need. For instance, an elementary geometrical proof can be given of the fact that no rational number exists which is the ratio of diagonal to side of a square. More generally, if O and A are *given* points and P a *variable* point collinear with them, positions of P can, of course, be specified for all rational values of the ratio $OP : OA$*; but it is not, conversely, true that a position of P, chosen at random on the line, necessarily gives a rational number as "ratio $OP : OA$" (if, indeed, such a "ratio" exists in that case). The "denseness" of the Rational Number system ensures that we can get *as near as we like* to any chosen position of P by means of rational values for the ratio $OP : OA$; but it does *not* ensure that we shall actually get, in that way, *to* that position: we may or may not be able to do so.

This illustration, by points on a straight line, is in fact of the very essence of the question under consideration. The *continuity* of the straight line is something more than the denseness of points (P) determined—from two given points O and A on it—by rational values of $OP : OA$. And the Irrational Numbers are the "numbers" required to complete the relationship between the positions of P and the ratio $OP : OA$. Thus *the system of Real Numbers* (rational and irrational) *is more than "dense"; it is a "continuous" system.*

(ii.) This approach to the Irrational Numbers, by the principle of Ratio, does not establish the existence of such "numbers"; but it indicates their significance if we take their mathematical existence for granted, in advance. It arrives at these numbers by a kind of instinct or intuition—the natural way, and the actual historical way, of arriving at most of the important fundamental facts of mathematical science.†

* When the ratio is *positive*, the direction of OP is the *same* as that of OA; when *negative*, the two directions are *opposite* [See Note at the end of Ch. IV., p. 28].

† Practically all elementary knowledge of the Infinitesimal Calculus (of which the subject-matter is the continuity of the Real Number system) has been built on this particular intuition—facilitated by the device of equating number-quantities to geometrical quantities: for instance, $x = \text{OM}$, $y = f(x) = \text{MP}$

The mathematical justification for this new use of the term "number" turns upon the possibility of defining operations of Addition, Subtraction, Multiplication and Division, to correspond—and operations of the other three kinds, when we come to regard them as independent of these. This is where we bring the third principle of § 19 into play.

(iii.) On the line OA, used in (i.), a point chosen at random is *either* (1) a position of P for which $OP : OA$ is a rational number, *or* (2) it must separate *all* the points so related to the rational numbers into two classes—those to one side of it and those to the other.

Hence in case (2) we have—instead of the rational number $OP : OA$—*a separation of all the Rational Numbers* into two classes, say* C_1 *and* C_2, *such that those of the class* C_1 *are all less than those of the class* C_2; and this separation is, on account of the "denseness" of the "Rational Points" on the line, such that *there is neither a greatest rational number of the class* C_1 *nor a least of the class* C_2. (A terminal of either class would, by hypothesis, give a point *to one side or other* of the position of P in question—that is, *at a finite distance* from it—and therefore such that an infinity of "rational points" could be located between: "which is absurd.")

As such separations of the "Rational Points" on the line OA are determined by points which are not themselves "rational," so (conversely) *such separations of the Rational Numbers are used to determine the Irrational Numbers*; and it is in terms of them that operations of Addition and Multiplication are defined, and inversely of Subtraction and Division—primarily for *positive* "real" numbers (rational or irrational), and then (by an obvious transition) for all the Real Numbers. But the details of the processes are neither simple nor elementary; and the whole subject would therefore have to be regarded as beyond the range of elementary discussion but for the following facts :—

(iv.) (1) The processes of Decimal Approximation discussed in §17 are, in fact, systematic processes of separating

* The term "separation" is used in this book as the most satisfactory equivalent of Dedekind's term *schnitt*.

For a clear conception of "all the Rational Numbers," the reader is recommended to have in mind the tabulation of Ch. IV., §15, (viii.).

all the *Decimal* Numbers into classes which may be described, *for these numbers*, exactly as the classes C_1 and C_2 are described in (iii.). For example, when we write

$$(N =)\ 747/35 = 21{\cdot}3\dot{4}2857\dot{1}$$

we mean that

$21 < N < 22$; $21{\cdot}3 < N < 21{\cdot}4$; $21{\cdot}34 < N < 21{\cdot}35$; . . . .
. . . $21{\cdot}3428571428 < N < 21{\cdot}3428571429$; . . . . . .

—separating first the integral numbers ; then the decimal numbers of denominator 10 ; and so on. And it is possible to say, of any specified decimal number, whether it belongs to the C_1 class or to the C_2 class.

This means that the use of decimal *approximations* in algebraic operations can equally well be regarded as the use of decimal "*separations*" ; and it therefore means that we have some actual experience upon which to build the more general ideas required for irrational number theory.

In this connection it is most important to study closely the use of decimal approximations in operations of addition, subtraction, multiplication and division—more especially of the two "inverse" operations—with a view to determining *the degree of approximation of the results.**

(2) Still more to the point is the fact that *the Irrational Numbers can be determined just as precisely by separations of the Decimal Numbers as by separations of all the Rational Numbers.*

It is the denseness of the separated system that is the determining circumstance ; and the dense sub-system is just as good, for this purpose, as the dense system of which it is a part. The proof consists in showing that two different separations of the Rational Numbers (giving different irrational numbers) imply two different separations, also, of the Decimal Numbers—the converse being obviously true.

The "irrational" separations of the Rational Numbers have, as essential characteristic [See (iii.) above], no terminal rationals

* The best general guide is commonsense, in applying the most elementary principles of Inequality. See also "A New Algebra," by Barnard and Child, Vol. II., Ch. XXVIII. (and following chapters).

at the point of separation. But, while "irrational" separations of the Decimal Numbers have also this characteristic, it is not conversely true that Decimal separations with this characteristic necessarily give irrational numbers. They may give *non-decimal* rational numbers [See Ch. IV., § 17 (ii.)]. We know that in that case the decimal expressions "recur"; and that is an essential distinction between rational and irrational numbers; but it is a distinction of a secondary kind, which has not the same theoretical importance as the fundamental distinction between *all* the Rational Numbers and *all* the Irrational Numbers.

(v.) The important point, however, for the present discussion is that "*Decimal Approximation*" *is just as applicable to the Irrational Numbers as it is to the Rational Numbers.*

It is this very practical fact which makes the transition, from the comparatively elementary Rational Numbers to the theoretically abstruse Irrational Numbers, relatively easy—so easy, in fact, that the step is commonly taken without any consciousness at all of the essential underlying difficulty.

Just because of the ease of this transition, however, it seems important to emphasise the fact that "approximation" is without meaning except by reference to something "exact," *to* (or towards) which one is approximating. It is essential to accurate mathematical thought to realise that an irrational number is just as precise and definite a "number" as an integral or a fractional number. This precision is, of course, most clearly realised by reference to the points on a straight line OA, which correspond indifferently to the Rational and the Irrational Numbers.

(vi.) We shall proceed on the assumption that enough has been stated to explain the "existence" of the system of Real Numbers, Rational and Irrational, subject to properly defined operations of Addition, Subtraction, Multiplication and Division—with their Laws of Commutation, Association and Distribution.

The actual definitions of the four Rational Operations as applied to the Real Numbers, in general, are precise mathematical forms—involving some considerable difficulties in detail—corresponding to the common-sense uses of decimal approximations to which reference has been made in (iv.), (i) above.

All practical needs are met by the fact that decimal approximations, to any required degree of accuracy, can always be obtained—and used (with error of known order) in place of the real numbers actually in question.

21. (i.) In terms of irrational numbers we can now take the main step in the removal of restrictions from the operation of Evolution.

If a positive rational number, a, has not a rational n^{th} root—as, for example, $+47$, or $+47/6$, has not a rational 13^{th} root—a separation of *all* the *positive* rational numbers can be made, into those (the class C_1) of which the n^{th} powers are less, and those (the class C_2) of which the n^{th} powers are greater than a; that is to say, it can be determined of *any* named positive rational number whether it is of the class C_1 or of the class C_2. And this separation gives a positive *irrational* number, which can be *proved* (by multiplication*) to have its n^{th} power *equal* to a.

Thus Evolution, denoted by $\sqrt[n]{a}$†, is defined, in terms of positive real numbers, for *positive* rational values of a. And we note, in passing, that if n is *even*, the *opposite* negative real number is also such that *its* n^{th} power $= a$ (by the laws of sign, in Multiplication). If a is *negative*, we have (by these same laws) a *negative* real n^{th} root if n is *odd*; but if n is *even*, there is *no real n^{th} root*. This case of $\sqrt[n]{a}$, when a is negative and n even, remains as an important restriction upon the operation. We shall find that the removal of it brings us to the final stage of the Number theory; and we shall find that the other side-issues just mentioned are more nearly related to it than to the primary case. [See Ch. VI., § 27, (i.), (3).]

Taking the simplest cases of Evolution, we have—
$\sqrt[n]{(+1)} = +1$, for all values of n (*and* -1, when n is even);
$\sqrt[n]{(+2)}$ is irrational for all the values 2, 3, 4, · · · · of n;

* Proofs of such propositions are essentially difficult, because of the essential difficulty of the exact theory of the operations with irrational numbers.

† $\sqrt[n]{\ }$ is best thought of as a *composite* symbol ($\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, · · ·). If $\sqrt[n]{a} = b$, then $a = b^{+n}$; for example, if $\sqrt[5]{a} = b$, $a = b^{+5}$. As we shall see, *it is never generalised beyond this case.* [§ 22, (iii.)].

so, $\sqrt[n]{(+3)}$; $\sqrt[2]{(+4)} = \pm 2$, but $\sqrt[n]{(+4)}$ otherwise irrational; $\sqrt[n]{(+5)}$ always irrational; and so on. The irrational numbers of this kind are commonly called "surd-numbers"; but the term "surd" appears to have been over-emphasised in Elementary Algebra; it is a survival of days when these numbers were not realised as "part and parcel" of a more general system.*

There are a variety of possible ways of determining the Decimal separations for $\sqrt[n]{a}$, such as $\sqrt[2]{3} = 1{\cdot}732 \cdots\cdots$ (for *square roots*, the process of § 11, (iii) may be adapted); but in practice the general problem is reducible to one of logarithms, of which the tabulation has very great and far-reaching practical importance. The operation of Logarithmation, in fact, quite overshadows—though it does not altogether supersede—the operation of Evolution, in the final developments of the Number theory. [See §§22-3].

(ii.) The principles applied to the determination of $\sqrt[n]{a}$, when a is rational, apply equally well when a is irrational; for example, $\sqrt[3]{\sqrt[5]{(+7)}}$ is a perfectly definite irrational number. And, in fact, *a positive "real" value for $\sqrt[n]{a}$ always exists when a is "real" and positive.*

(iii.) The following Evolution theorems—for which it is sufficient (as a foundation) to confine ourselves to *positive real quantities throughout*—are corollaries† to foregoing principles:—

(1) $\sqrt[n]{(a \times b)} = \sqrt[n]{a} \times \sqrt[n]{b}$ and $\sqrt[n]{(a \div b)} = \sqrt[n]{a} \div \sqrt[n]{b}$

for example, $\sqrt[3]{(+10)} = \sqrt[3]{(+2)} \times \sqrt[3]{(+5)}$,

and $\sqrt[3]{(+2/5)} = \sqrt[3]{(+2)}/\sqrt[3]{(+5)}$.

(2) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[p]{a} = \sqrt[n]{\sqrt[m]{a}}$, if $p = m.n = n.m$;

for example, $\sqrt[3]{\sqrt[5]{(+7)}} = \sqrt[15]{(+7)} = \sqrt[5]{\sqrt[3]{(+7)}}$.

(The equality of n^{th} powers in case (1), of p^{th} powers in case (2), is *sufficient* for the truth of these propositions—*since*

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* Surd-numbers belong to an important sub-system of the Real Numbers, called "the Algebraic Numbers"—being all the real numbers which can be specified as roots of "Algebraic Equations" *i.e.*, equations of the type $a_n.x^n + a_{n-1}.x^{n-1} + \cdots\cdots + a_1.x + a_0 = 0$, in which the coefficients are integral.

† These corollaries raise again, of course, the fundamental difficulties of § 20.

these quantities have each, in case (1), *only one positive* n^{th} *root*,—and so on).

22. (i.) The generalisation of Evolution enables us to extend *Involution* to *fractional exponents*.

(1) For this purpose we use the identity $a^{\pm m/n} = \sqrt[n]{a^{\pm m}}$, which we know to be true if $\pm m/n$ is a *fractional expression for an integral number* [See §§ 9, 14 (iv.), 18]; for example, $(47/6)^{-3} = (47/6)^{-12/4} = \sqrt[4]{(47/6)^{-12}} = \sqrt[4]{(6/47)^{+12}}$. Using now the further fact that the expression $\sqrt[n]{a^{\pm m}}$ has been *generally* defined (§ 21) for positive real values of a (the power of a involved in it having a positive or negative *integral* exponent denoted by $\pm m$), we *define* $a^{\pm m/n}$, *when the exponent is actually a fractional number, to have a positive real value given by* $\sqrt[n]{a^{\pm m}}$ *when* a *itself is real and positive.*

(2) It is easy to see that all the fractional expressions for a given rational exponent, r, (integral or fractional) yield the same positive real value for a^r; for $a^{\pm k.m/k.n} = \sqrt[k.n]{(a^{\pm k.m})} = \sqrt[n]{\sqrt[k]{(a^{\pm m})^{+k}}} = \sqrt[n]{a^{\pm m}}$—see §§ 18 and 21, (iii.)—k, m, n denoting representative natural number "figures" [See foot-note to § 21 (i.), *p.* 36, on the composite symbol $\sqrt[n]{\ }$].

(3) Further $(\sqrt[n]{a})^{\pm m}$ yields the same positive real value; for its n^{th} power $= ((\sqrt[n]{a})^{+n})^{\pm m}$—see § 18—that is, $a^{\pm m}$; and there is only one positive real number of which this is true.

(ii.) It is then easy to *prove* that *for positive real values of the "bases" and positive real values of the "powers"*—but rational values (integral or fractional, positive or negative) of the "exponents"—the Involution theorems hold; namely,

(1) $a^{b+c} = a^b \times a^c$ and $a^{b-c} = a^b \div a^c$;

(2) $a^{b \times c} = (a^b)^c$ and $a^{b \div c} = a^{b \times c'} = (a^b)^{c'}$, if $c.c' = 1$;

(3) $(a \times b)^c = a^c \times b^c$ and $(a \div b)^c = a^c \div b^c$.

The mode of proof is simple, as follows :—If $\pm$ M/N be the reduced fractional form of (1) $b \pm c$ or (2) $b \overset{\times}{\div} c$ or (3) c, as the case may be, the relation, say P = Q, holds if $P^{+N} = Q^{+N}$ holds; and this latter is, in each case, reducible to the theorems of § 18 (on Involution with *integral exponents*). For example, if $b = +2/3$ and $c = -5/7$,

$$a^{b \div c} = a^{-14/15} = \text{P} \quad \text{and} \quad (a^b)^{c'} = (a^{+2/3})^{-7/5} = \text{Q};$$

therefore, $\text{P}^{+15} = a^{-14}$, and

$$\text{Q}^{+15} = (\text{Q}^{+5})^{+3} = ((a^{+2/3})^{-7})^{+3} = ((a^{+2/3})^{+3})^{-7} = (a^{+2})^{-7} = a^{-14}.$$

In this example, the original numerators and denominators (of b and c) have been taken "prime." The modifications which naturally enter when "cancelling" occurs are quite simple.

It cannot be too strongly emphasised that these proofs, and the theory of this entire section, turn on the restriction of " bases" and " powers" (but not " exponents") *to be positive quantities.* This restriction ensures that the relation between base and power is, in each case, a " one-one relation"—that is, given values for base and exponent, there is one and only one value (in question) for the power; and, conversely, given values for power and exponent there is one and only one value (in question) for the base (or " root). If this restriction is strictly adhered to at this stage, the general discussion of these operations at a later stage—when we are in a position to discuss them with complete generality [See Ch. VI., § 27]—is greatly simplified.*

(iii.) Since $\sqrt[n]{a} = a^{+1/n}$, by the definition of (i), (1)—with its reference to positive real quantities only—it follows that the *original* operation of Evolution is included in the generalised operation of Involution. That is to say, the inverse (Evolution) of Involution with a positive integral exponent may now be *expressed* as another involution, with the " reciprocal" exponent. (But, of course, when we ask what " involution" *means* in these two cases, the answer is, in the first case, a special multiplication, in the second case, the evolution which is *defined* as the operation inverse to that fundamental type of involution).

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* In the final generalisation, $a^{\pm m/n}$ has n different values (nearly all " unreal"); but the *one* positive real value here taken account of, when a is real and positive, will be found basic to the more general treatment.

We proceed to show that for the case of the *general* rational exponent, the evolution-type of inverse of an involution is another involution—with the reciprocal exponent :—

If $a^{\pm m/n} = b,$ then $b = \sqrt[n]{a^{\pm m}}$

$\therefore\ b^{+n} = a^{\pm m}$ (that is, a^{+m} or $1/a^{+m}$).

Hence $a^{+m} = b^{+n}$ or $1/b^{+n}$; that is, $a^{+m} = b^{\pm n}$

$\therefore\ a = \sqrt[m]{b^{\pm n}} = b^{\pm n/m}$

Thus $a^r = b$ and $a = b^{r'}$

are equivalent relations, if r, r' are rational and such that $r \times r' = 1$, and if a and b are both restricted (for the moment) to be real and positive. For example, $a^{-8/13} = b$ and $a = b^{-13/8}$ are equivalent in that sense.

(iv.) To sum up: (1) An involution with a fractional exponent is defined by two more elementary operations (in either order, at this stage)—namely, an involution with an integral exponent, and an evolution; and, in particular, if the exponent is the reciprocal of a positive integer the involution reduces simply to an evolution.

(2) Involution, thus generalised, includes its own evolution-type of inverse—reciprocal exponents giving mutually inverse operations (for example, $a^{-8/13}$ and $a^{-13/8}$ are mutually inverse forms—as a^{+3} and $\sqrt[3]{a}$ are). And *this absorption of Evolution in a generalised Involution makes it unnecessary* to generalise Evolution, as an independent operation.*

"Evolution" retains its original meaning, as denoted by $\sqrt{\ }, \sqrt[3]{\ }, \sqrt[4]{\ }, \ldots$, while all the other operations are generalised, *as such.*

23. We are now in a position to remove the main restrictions from the operation of Logarithmation.†

(i.) The relation $a^r = b$, as so far defined (for rational values of r), may be written in the logarithmic-inverse form $\log_a b = r$. But if positive real values are *given* for a and b, there is *not*, in general, a corresponding *rational* value of r.

* If it were necessary, it could, of course, easily be done. It is important to realise the strict scientific economy of the developments.

† The absorption of Evolution in the generalised Involution leaves Logarithmation as the only ultimately *independent* operation inverse to Involution.

For example, to take simple important cases, if $a = +10$ and $b = +2, +3, \cdot \cdot \cdot \cdot \cdot, +9$ there are not corresponding rational values of r.

We can, however, in such cases, separate all the Rational Numbers into two classes C_1 and C_2, with reference to a and b, in such a way that the rational numbers of one of these classes —when used as exponents—give "powers of a" which are less than b, while those of the other class give powers greater than b*. And the separation so arrived at determines *uniquely* an *irrational* number, α, which may properly be used to *define* $\log_a b$; that is, $\log_a b = \alpha$; and, inversely, $a^\alpha = b$ defines a^α, *in that connection*—but only as a *restricted* inverse form, derived from the generalised definition of Logarithmation. We have not yet arrived at a meaning for a^b when b has *any* given irrational value.

Thus Logarithmation, which up to this point has had practically no place in the mathematical system—having merely been defined as a *form* arising out of the Natural Number principles—has now been given very general definition,† and actually takes precedence of Involution at this stage. It comes, in fact, in a certain sense, to share with Addition and Multiplication the character of a "direct" and fundamental operation—as we shall see in what follows [See (iii.) below].

The principle of Decimal Approximation is, of course, applicable to the definition of $\log_a b$, and all such quantities (for positive values of a and b) can be expressed decimally to any required degree of accuracy; for example,

$$\log_{10} 3 = \cdot 477121 \cdot \cdot \cdot \cdot$$

(ii.) The Logarithmation theorems—

(1) $\log_a(b \times c) = \log_a b + \log_a c$

and $\log_a(b \div c) = \log_a b - \log_a c$

(2) $\log_a b^c = c.\log_a b$ (c rational)

(3) $\log_a b = \log_c b \times \log_a c$, etc.,

in particular, $\log_a b \times \log_b a = 1$

and, therefore, $\log_a b = \log_c b / \log_c a$

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* As r increases by rational values, a^r increases if $a > +1$ (decreases, if $0 < a < +1$). See Appendix IV.

† A single real value of $\log_a b$ being determined by given *positive* real values of a and b.

—can then be proved, *with all the generality which the definitions, so far given, make possible*—each logarithm-quantity having *one and only one* real value. The proofs have, once again [See §21, (i.), footnote], the difficulties in detail of all fundamental irrational number work.

We note, before proceeding to consider the great practical importance of these theorems, that (3) in its final form permits reduction of all logarithms to logarithms with a standard base. Two particular bases have special claims to be used as such standards. One of these [see (iii.) below] is the "radix" number, 10, of the Decimal Notation; its *practical* usefulness is of the very greatest importance. The other is an irrational number—generally denoted by e—which emerges naturally from the theory at a more advanced stage; see Ch. VI., § 27, (ii.).*

$$[e = 2{\cdot}7182818285 \cdot \;\cdot\;\cdot\;\cdot\;\cdot]$$

(iii.) (1) The peculiar importance of the Decimal Numbers —with their specially simple scheme of notation—makes "logarithms to base 10" play a very important part in Mathematics.

Multiplication of a decimal number by a power of 10 with an integral exponent (positive or negative) does not affect the digits of its decimal expression (which we may suppose —see end of § 10, (ii.), (3), and *footnote* to § 16—to have as many noughts as we please at either end), but merely affects the position of the decimal point relative to these digits; multiplication by 10 moves the decimal point one place to the right (division, one to the left). Hence by (ii.), (1) a set of numbers so related have logarithms to base 10 which *differ* integrally.

Thus the logarithms to base 10 of all (other) decimal numbers have a simple relationship to those of the decimal numbers between $+1$ and $+10$. These latter have logarithms between 0 and $+1$.†

* It is interesting to compare with the fact that in angle-theory there are *two* angles which have special claims to use as standard *units*; one, the whole-plane angle (or the straight angle, or the right angle), of great practical importance; the other, the radian, arising naturally from more advanced theoretical considerations.

† Since $\log_{10} x$ increases with x. See Appendix IV.

Hence the logarithm to base 10 of a decimal number is expressed as the sum of two parts. One, called "the characteristic," is an integral number (*positive or negative*) which can be written down by inspection from the position of the decimal point—being determined by the number of places by which that point is to the right or left of the standard position (of zero characteristic). The other, called "the mantissa," is the logarithm of a number between + 1 and + 10. For example,

$$\log_{10}342{\cdot}0716 = 2 + \log_{10}3{\cdot}420716 = 2{\cdot}534117 \quad \text{(approx.)}$$

$$\log_{10}{\cdot}003420716 = -3 + \log_{10}3{\cdot}420716;$$

$$= -3 + {\cdot}534117 \quad \text{(approx.)},$$

$$\text{written } \bar{3}{\cdot}534117$$

(2) Tables of approximations to the mantissa quantities, for the logarithms of decimal numbers at regular intervals, are the practical means by which nearly all the (approximate) arithmetical calculations of Higher Mathematics are performed. Such Tables vary in respect of the interval, and of the degree of approximation to the mantissae. "Four-figure Tables" give approximations to $\log_{10} 1{\cdot}001$, $\log_{10} 1{\cdot}002$, . . . $\log_{10}9{\cdot}999$—approximations which are (or should be) accurate to four places of decimals. These are very convenient, but for many purposes not accurate enough. From the more accurate Tables the logarithms of numbers between those of which the logarithms are (approximately) tabulated can be approximately determined, by assuming *proportional* increase through each "gap"—an assumption which is more and more justified the smaller the interval of the Table. The ideal Table will clearly be one for which the interval* is such, and the degree of approximation such, that the proportionality in question gives approximations of the same degree of accuracy as those actually tabulated.

The actual determination of these approximations, for tabulation, involves theoretical considerations which are beyond the scope of this discussion. But *use* of the Tables is quite elementary.

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* In such a Table different intervals would, with advantage, be used at different parts of the tabulation.

The Logarithm Table shares with the Addition Table and the Multiplication Table a place of basic importance in the processes of arithmetical calculation.

(iv.) Even multiplications and divisions are commonly performed (approximately) by the help of Logarithm Tables, using (ii.), (1).

Involutions with integral exponents are almost invariably performed by means of (ii.), (2). There is practically no other way of performing evolutions—and involutions with rational exponents.

For these practical arithmetical processes, since they involve the calculating of a logarithm and then obtaining the number of which it is the logarithm, a Table of "anti-logarithms" for the latter step is more convenient—because more systematic—than inverse use of the Logarithm Table. This is, of course, an Involution Table, giving the (approximate) values of 10^r for decimal values (between 0 and +1) of r at the regular interval of the Table.

For example, [See § 21, (i.)].

$$\log_{10}\sqrt[13]{47} = \log_{10}47^{1/13} = (1/13).\log_{10}47$$
$$= 1{\cdot}6721/13 \text{ (approx.)}$$
$$= 0{\cdot}1286 \text{ (approx.)}$$

using Four-Figure Log. Table.

(If the divisor were a much larger number, and/or more accurate Tables were used, it might save time to make a second application of the Log.—and Anti-Log.—Tables to evaluate the quotient—introducing, of course, a second source of inaccuracy).

Hence $\sqrt[13]{47} = 1{\cdot}345$ (approx.), from the Anti-Log. Table.

24. (i.) Involution can now be extended to *all* irrational exponents.

Using the facts of § 23, a^b, when a is real and positive and b irrational (positive or negative), may be a positive rational number, c, such that $\log_a c = b$ [§ 23, (i.)]. But, if not, then all the positive rational numbers can be separated—with reference to a and b—into two classes, C_1 and C_2, such that the numbers of one of these classes give logarithms, to base a, which are less than b, while those of the other class give logarithms greater than b.* And this separation determines a positive irrational number, α, which can then be *proved*† such that $\log_a \alpha = b$—and, therefore, $a^b = \alpha$.

Thus Involution is theoretically established for positive real values of the base and *all* real values of the exponent. (From the practical point of view, of course, the evaluation of such "powers" is not distinguished from the case in which the exponent is rational. The same process of Decimal Approximation [See § 23, (iv.)] applies to both. But the theoretical distinction is theoretically important).

(ii.) The Involution theorems can then be proved, with corresponding generality, simply as converses of the Log. theorems of § 23 (ii.).‡ The theorems are—

(1) $a^{b+c} = a^b \times a^c$ and $a^{b-c} = a^b \div a^c$

(2) $a^{b \times c} = (a^b)^c = (a^c)^b$ [and $a^{b \div c} = a^{b \times c'}$ if $c.c' = 1$]

(3) $(a \times b)^c = a^c \times b^c$ and $(a \div b)^c = a^c \div b^c$

in which each *base* and each *power* is understood to be real and positive. The proof that $P = Q$ in each of these cases consists in shewing that $\log P = \log Q$ — the base of these logarithms being, most simply, a in cases (1) and (2), any base in case (3). (*Note.*—The simplicity of the proof turns—here, as in § 22, (ii.)—on the *uniqueness* of the relationship between —or the one-one correspondence of—x and $\log_a x$, when a is given and x varies, under the conditions expressly imposed at this stage of the theory).

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* As r increases by positive rational values, $\log_a r$ increases if $a > +1$ (decreases, if $0 < a < +1$). See Appendix IV.

† The proof has the characteristic difficulties of fundamental work on irrationals. [See § 21 (i.) and 23 (ii.)].

‡ The proofs require, as a preliminary, the extension of § 23, (ii.), (2) to the case of the general real exponent—as follows :—
If $b^c = p$, then, by (i.) above, $c = \log_b p$
$\therefore$ $c = \log_a p / \log_a b$, by § 23, (ii.), (3), or $\log_a p = c. \log_a b$.

CHAPTER VI.

THE COMPLEX NUMBERS.

25. (i.) The final development of the Number system is directly related to the outstanding restrictions on Evolution and Logarithmation (and Involution).

The Evolution expression, $\sqrt[n]{a}$, has not been defined for a negative and n even; for example, $\sqrt[4]{(-5)}$. And the second real "root" when a is positive and n even has been deliberately excluded from the developments of the immediately preceding sections—so as to preserve a certain important element of simplicity* (the one-one correspondence between base and power, for a given exponent). This restriction is an essential condition of the generalisation, thus far made, of Involution and Logarithmation.

(ii.) It is easy to see that the primary and fundamental case of Evolution, for further extension of that operation, is that of $\sqrt[2]{(-1)}$. If the development is to be of any mathematical importance we must be able to analyse thus:—

$$\sqrt[n]{a} = \sqrt[n]{(a' \times (-1))} = \sqrt[n]{a'} \times \sqrt[n]{(-1)}, \text{ if } a = -a'$$
$$\text{and} \quad \sqrt[n]{(-1)} = \sqrt[m]{\sqrt[2]{(-1)}} \quad , \text{ if } n = 2.m,$$

reducing to $\sqrt[2]{(-1)}$ as the primary problem.

We therefore proceed by *postulating* a "number," denoted by ι (and called "iota"), such that $\iota^2 = -1$ —in the not ill-founded hope (based on foregoing experience) that we shall find this number to have an actual "existence," similar to that of the Negative Integral Numbers, the Fractional Numbers and the Irrational Numbers. (We shall, in fact, find that the principle of Ratio is equal to this further demand upon it. The possibilities of Direction,† as related to Positive

* There is a certain analogy between this carefully safeguarded simplicity, in relation to subsequent developments, and that of the originally conserved simplicity of the Natural Numbers.

† See Note at end of Ch. IV., p. 28.

and Negative Numbers, are not yet exhausted ; and they can be used to give the necessary extension to the conception of Ratio. But by far the greatest importance of the Complex Numbers is not their direct practical application to measurement of quantities, but the fact that they provide for unrestricted use of the Number Operations).

(iii.) The first requirement of the inclusion in the Number-system of such a new number as ι is that it shall be capable of use, along with the Real Numbers, in the operations which are already fully established—namely, the Rational Operations (Addition, · · · · Division). This involves the further postulation of new types of "number," as follows :—

(1) Beginning with Multiplication,* to which ι is specially related (by the defining property $\iota \times \iota = -1$), we have to postulate "numbers" represented by $\iota \times x$ and $x \times \iota$ (defined to be equal) for all real values of x. It is to these that the name "Imaginary† Numbers" has been given. They may be regarded as the numbers (if we assume them to exist) of which the "squares" are *negative* real numbers. And they themselves may be qualified as "positive" or "negative," according as x has positive or negative real values.

On the general principle that multiplication by $(+1)$ gives product equal to number so multiplied, the "imaginary number" $(+1).\iota$, written $+\iota$, must be taken as the same number as the original ι of (ii.). Thus ι becomes what we may call "the leading *imaginary* number."

But since this basic "number" of the new system was characterised solely by the property that its "square" is -1, and since that property is shared by the *opposite* "imaginary number" $(-1).\iota$, or $-\iota$, we see that if we were to replace the original ι by ι', and take this latter "number" to be $-\iota$, the subsequent *theory* would be unaffected. We have thus the following important principle of Unreal Number theory :—

In any process, or relation, involving ι, we may replace ι, throughout, by $-\iota$, without affecting the validity.

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* See § 15 (iv.), p. 23 *footnote.*

† Attention has already been directed to the inadequacy of these technical terms. See Note to § 12 (Ch. II.).

The further general principle that multiplication by 0 gives product also 0, leads to the *definition* $\iota \times 0 = 0 \times \iota = 0$; hence to the fact that 0, which is the connecting link between Positive and Negative Numbers, is also the connecting link between Real and Unreal Numbers.

The principles of Association and Commutation, established as characteristic of Multiplication, yield the further *definitions* that *the product of a real number and an imaginary number is an imaginary number* $[x.(y.\iota) = (x.y).\iota]$; and *the product of two imaginary numbers is a real number* $[(x.\iota).(y.\iota) = x.y.\iota^2 = x.y.(-1) = -x.y.]$.

In particular, we have

$$\iota^3 = \iota^2 \times \iota = -\iota; \quad \iota^4 = (-1)^2 = +1; \quad \iota^5 = \iota; \quad \text{etc.}$$

(2) For Division, we have simply the inverse facts.

Division of an imaginary number by a real number gives an imaginary number $[x.\iota/y = (x/y).\iota]$; so, again, of a real number by an imaginary number,

$$[x/(y.\iota) = x \div (y \times (-1) \div \iota) = x \div y \div (-1) \times \iota = (-x/y).\iota];$$

and division of an imaginary number by an imaginary number gives a real number $[(x.\iota)/y.\iota) = x/y]$.

In particular, division by $\pm\iota$ is equivalent to multiplication by $\mp\iota$.

(3) For Addition and Subtraction, the principle of Distribution requires *definition of the sum and difference of two imaginary numbers to be imaginary numbers* $[x.\iota \pm y.\iota = (x \pm y).\iota]$.

But for the sum and difference of a real number and an imaginary number it is necessary to postulate a further type of "numbers," represented by $x + y.\iota$, or $x + \iota.y$, for real values of x and y. It is for these that the name "Complex Numbers" is required. But this term is used for *all* the numbers represented by $x + \iota.y$, when x, y are real, and therefore covers, in particular, both the Real Numbers $[y = 0]$ and the Imaginary Numbers $[x = 0]$.

The Complex Numbers (when their existence has been established) are the general numbers of the mathematical system. The Unreal Numbers are, of course, the complex numbers which are not real—including, in particular, the Imaginary Numbers.

(4) Any change of either x or y (or both) implies a change from one complex number to another. [Compare Ch. IV., § 15, (iii.) on the *different* fractional expressions for a *given* rational number.]

For, if $x_1 + \iota.y_1 = x_2 + \iota.y_2$, then $x_1 - x_2 = \iota.(y_2 - y_1)$
$\therefore (x - x_2)^2 = -(y_1 - y_2)^2$ or $(x_1 - x_2)^2 + (y_1 - y_2)^2 = 0$
which can only be true (of real x, y quantities) if $x_1 = x_2$ and $y_1 = y_2$.

(iv.) (1) In Addition and Subtraction, complex numbers may clearly be defined to give complex results

$$[(x_1 + \iota.y_1) \pm (x_2 + \iota.y_2) = (x_1 \pm x_2) + \iota.(y_1 \pm y_2)]$$

(2) For Multiplication, we have, by these same principles,

$$(x_1 + \iota.y_1).(x_2 + \iota.y_2)$$
$$= x_1.x_2 + \iota.(x_1.y_2 + x_2.y_1) + \iota^2.y_1.y_2$$
$$= (x_1.x_2 - y_1.y_2) + \iota.(x_1.y_2 + x_2.y_1)$$

which gives the means of defining this operation, also, in "complex" terms.

In particular, $(x + \iota.y).(x - \iota.y) = x^2 + y^2$
and this "real" product of "*conjugate* complex numbers" gives the obvious means of transition to Division.

(3) Thus for Division we have,

$$(x_1 + \iota.y_1)/(x_2 + \iota.y_2)$$
$$= (x_1 + \iota.y_1).(x_2 - \iota.y_2)/(x_2^2 + y_2^2)$$
$$= (x_1.x_2 + y_1.y_2)/(x_2^2 + y_2^2) - \iota.(x_1.y_2 - x_2.y_1)/(x_2^2 + y_2^2)$$

giving the means of defining this operation, too, in "complex" terms.

(v.) (1) But, for Multiplication and Division, a simple transformation—familiar in Elementary Trigonometry, and still more familiar in Analytical Geometry—produces a striking simplification, which proves to be of the utmost importance in the final stages of the Number theory.

The transformation in question is

$$x + \iota . y = r . (\cos\theta + \iota . \sin\theta)$$

from the substitution $x = r.\cos\theta$, $y = r.\sin\theta$

whence $r^2 = x^2 + y^2$, $(\cos\theta, \sin\theta) = (x, y)/r$

(It is the transformation from Cartesian to Polar co-ordinates, in Analytical Geometry).

The simplification resulting from this transformation is a consequence of the *Demoivre identity*, namely,

$$(\cos\theta_1 + \iota.\sin\theta_1).(\cos\theta_2 + \iota.\sin\theta_2)$$
$$= \cos(\theta_1 + \theta_2) + \iota.\sin(\theta_1 + \theta_2);$$

in particular,

$$(\cos\theta + \iota.\sin\theta).(\cos\theta - \iota.\sin\theta)[= \cos^2\theta + \sin^2\theta] = 1.$$

The function of θ specified by $\cos\theta + \iota.\sin\theta$ is conveniently denoted by cis θ. Thus $\text{cis}\,\theta.\text{cis}(-\theta) = 1$

(2) If we write $z = x + \iota.y = r.\text{cis}\,\theta$, denoting the "complex variable" by z, and if we take the real quantity r to be *positive**—for reasons following upon those for the restrictions to positive real numbers imposed in §§ 21-4—r is called "the modulus of z," and written mod z or $|z|$; and the many-valued θ—defined as *circular measure*—determined by $\cos\theta = x/r$, $\sin\theta = y/r$, is called "the Amplitude of z," and written Amp z. A one-valued amplitude, denoted by θ_0 and written amp z, can be defined by introducing a restriction to any continuous range of extent $2.\pi$; the range taken is bounded by $\pm\pi$, *one* of the two extremes being excluded (It does not seem to matter which). This amp z is said to give "the principal value" of the Amplitude. The value is 0 for a positive real number, $\pm\pi$ for a negative real number, $+\pi/2$ for a positive imaginary number, $-\pi/2$ for a negative imaginary number.

* Lest it be thought, by readers conversant with Analytical Geometry, that the reference to Polar Co-ordinates *implies* a positive r, it may here be remarked that the reasons for the restriction are much more cogent in the present connection. See an article by the writer, in The Mathematical Gazette, July 1923, p. 330.

(3) From the expression $z = r. \text{cis}\,\theta$, we have

$$z_1.z_2 = r_1.r_2.\ \text{cis}\,(\theta_1 + \theta_2)\ ,\ z_1/z_2 = r_1/r_2.\ \text{cis}\,(\theta_1 - \theta_2).$$

Thus, modulus of product, or quotient

= product, or quotient, of moduli

Amplitude of product, or quotient

= sum, or difference, of Amplitudes

(the latter being true of the many-valued "Amplitude," not necessarily of the one-valued "amplitude.").

Note the simplicity of these forms, as compared with those of (iv.), (2) and (3).

(vi.) From the facts of (iv.) and (v.), it is clear that the Laws of Association, Commutation and Distribution—for the Rational Operations—will apply, in their general forms [See Ch. I, § § 3, 5, 6], to "the complex numbers."

26. (i.) The discussion of § 25 has necessarily been of a tentative kind: a search after "numbers" for which a certain theoretical need is realised, but without that direct practical intuition which played so important a part in the Real Number development. (A difference reflected in the terms "real," "imaginary," "unreal.")

The actual establishing of the new numbers in the mathematical system is a process of the same two-fold kind as that used at previous stages—with somewhat different emphasis, as indeed the emphasis differed at these. [See § 15 and § 20]. We shew first that the geometrical principles of § 20 have a natural extension for this purpose, based on the facts of §25, (v.); and we then proceed to shew that a complete system of operations—properly generalised—is applicable to the new numbers (made "tangible" by the geometrical argument). The present section will be confined to the first of these purposes; the second has some considerable theoretical difficulties, which cannot be regarded as elementary, and the discussion of it, in the following section, will therefore be to some extent summary and incomplete.

(ii.) (1) The importance of the transformation of § 25 (v.) suggests a relation of the Complex Numbers to the *points* of a Plane system determined by "Rectangular Cartesian co-ordinates" x, y and "Polar co-ordinates" r, θ —with reference to a standard "origin" O in the plane and a standard "initial line" OX* (or OA, if A be the point of the line corresponding to $x = +1$, as in § 20).

The relation between points and real numbers, being given by OP : OA $= x$, for positions of P on the x-axis, may then properly be generalised to OP : OA $= z$, for *all* the points of the Plane-system—the quantities OA, OP being "vector quantities," of which the *lengths* have ratio r, or $|z|$, while the inclination of the *direction* OP to the *direction* OA is the angle given by θ, or Amp z. And z may therefore be called the "measure" of the vector OP in terms of the unit-vector OA.

(2) This extension to line-vectors of the conception of Ratio is in agreement with the fundamental principle of Vector-Addition, namely, PQ + QR = PR, for any three positions of the points P, Q, R (or, again, OP + OQ = OS, if OPSQ be a parallelogram). For,

$$z_1 + z_2 = (x_1 + x_2) + \iota.(y_1 + y_2)$$

and $x_1 + x_2$, $y_1 + y_2$ are the Cartesian co-ordinates of the point P′ such that OP_1 P′ P_2 is a parallelogram. Hence *the measure of the vector-sum* OP′

= *the sum of the measures of* OP_1 *and* OP_2

—a generalisation of one of the standard Ratio-theorems.†

In particular, $z = x + \iota.y$ corresponds to the vector-relation OP = OM + MP = OM + ON, if PM, PN be the perpendiculars from P to the x-axis and the y-axis, respectively; and the Imaginary Numbers correspond to the points (N) of

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* OY ⊥ OX, and ∠ XOY (or ∠ AOB, if we use the points on the unit-circle) is a *positive* rt ∠—this defining *either* the direction OY *or* the positive Trigonometric sense, according to order of procedure.

† A geometrical corollary, of great importance in higher theory, is the proposition that—

modulus of sum < (or =) sum of moduli of terms.

The algebraic proof is an interesting exercise in elementary inequality work.

the y-axis. The axes of reference may therefore be called, respectively, "the Real axis" and "the Imaginary axis" of the geometrical representation.

(3) The *ratio of any two line-vectors* of a Plane system may then clearly be *defined* as a complex number, namely, that which has the ratio of their lengths as modulus and the inclination of their directions as Amplitude-angle. And the relation

$$z_1/z_2 = (r_1/r_2).\ \text{cis}\,(\theta_1 - \theta_2)$$

may then be interpreted as the generalised Ratio-theorem—
*The ratio of two line-vectors = the quotient of their measures.**

These geometrical principles are clearly sufficient to put the (so-called) "Unreal Numbers" on much the same Number footing as the Real Numbers. And they have commonly been used, in some form, as the approach to the Unreal Numbers (by means of "the Argand Diagram"—the name originally given, on account of the originator, to the elementary figure of (ii.), in this connection). But this is over-emphasis of the geometrical aspect of the question; for the use of complex numbers in vector theory appears, in fact, to be relatively unimportant.† It puts the *theory* of these numbers upon a somewhat artificial basis. It is neither necessary nor desirable to have an essential geometrical reference in all uses of the Unreal Numbers.‡ [See Note on "Vector Analysis" at the end of this Chapter].

27. Returning to the Number processes—

(i.) (1) For Involution with integral exponents we have

$$z^{+n} = r^{+n}.\ \text{cis}\ n.\theta$$

and
$$z^{-n} = 1/z^{+n} = r^{-n}.\ \text{cis}\,(-n.\theta)$$

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* These generalised Ratio-theorems may then, of course, be applied to two vectors of any one kind, for example, *velocities* or *forces*, belonging to a given Plane system.

† For some interesting "Geometrical and Kinematical Illustrations" see a paper on the subject by Prof. R. W. Genese in the Mathematica Gazette, May 1923.

‡ The Complex Variable work in Alternating Current theory is commonly marred by an over-emphasis on the geometrical specification of complex numbers.

by § 25, (v.)—the different values of θ giving all, of course, the same value of the "power."

(2) For Evolution, if $z = w^{+n}$ and $w = \rho.\text{cis}\,\phi$, we have

$$r.\,\text{cis}\,\theta = \rho^{+n}.\,\text{cis}\,n.\phi \quad , \text{ by (1)}$$

$\therefore$ * $r = \rho^{+n}$, giving $\rho = \sqrt[n]{r}$ as defined in § 21,

and $\text{cis}\,\theta = \text{cis}\,n.\phi$, from which—on account of the many-valuedness of θ and ϕ—we have a number of different values of $\text{cis}\,\phi$, namely,

$$\text{cis}\theta_0/n,\ \text{cis}(\theta_0 \pm 2.\pi)/n,\ \text{cis}(\theta_0 \pm 4.\pi)/n,\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

It is easy to see that there are n *different* values of $\text{cis}\,\phi$ (which repeat themselves in "periods") and, therefore, n different values of w—all included in

$$z^{+1/n} = \sqrt[n]{r}.\,\text{cis}\,\theta/n.$$

(3) The relation just stated makes it convenient to distinguish between $\sqrt[n]{r}$ and $r^{+1/n}$, keeping the former for the positive real quantity (r being real and positive) defined in § 21, while the latter has n values given by

$$r^{+1/n} = \sqrt[n]{r}.\,\text{cis}\,2.k.\pi/n, \text{ for integral values of } k.$$

The n different quantities $\text{cis}\,(2.k.\pi/n)$ are clearly the n values of $(+1)^{+1/n}$; and since $\text{cis}\,(2.k.\pi/n) = (\text{cis}\,2.\pi/n)^k$, these n "n^{th} roots of unity" may be written,

$$\omega^0\ (= +1),\ \omega,\ \omega^2,\ \cdot\ \cdot\ \cdot\ \omega^{n-1}$$

if $\qquad \omega = \text{cis}\,2.\pi/n = \cos 2.\pi/n + \iota.\,\sin 2\pi/n.$

Again, $\quad \text{cis}\,\theta/n = \text{cis}(\theta_0 + 2.k.\pi)/n = \text{cis}\,\theta_0/n.\,\text{cis}\,(2.k.\pi/n)$

and $\therefore \quad z^{+1/n} = (\sqrt[n]{r}.\,\text{cis}\,\theta_0/n).(+1)^{+1/n}$

$$= [z^{+1/n}]\,.\,(+1)^{+1/n}$$

denoting by $[z^{+1/n}]$ the *one*-valued "power" obtained by using the principal amplitude, θ_0, of z (and said to give "the principal value of $z^{+1/n}$")—so that, in particular,

$$[r^{+1/n}] = \sqrt[n]{r}, \quad \text{and} \quad [(+1)^{+1/n}] = +1.$$

We note the special importance of the theory of the *single* positive real n^{th} root of a positive real number (§ 21), and how that theory, in its application to the modulus of the complex number in the general case, is basic to the more

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* See §25, (iii.), (4), and (v.), from which, if $z_1 = z_2$,

$$r_1 = r_2 \quad \text{and} \quad \text{cis}\,\theta_1 = \text{cis}\,\theta_2.$$

general theory. We see that it is not, in the particular case, the only n^{th} root, nor one of two (when n is even); it is one of n, of which the rest are all—or all but one—unreal. And we see, further, why it was properly regarded as being in a different category from the single negative real n^{th} root of a, when a is negative and n odd (a root which is *not* the "principal" n^{th} root in that case).

(4) It is convenient to insert here a note to the effect that the facts of (2) and (3) —expressed in the form that the equation $z^n = c$ has n different roots—may be regarded as a particular case of the important general theorem that *an algebraic equation of the n^{th} degree has n roots*—real or unreal, and not necessarily all unequal: more precisely, that a polynomial in z of degree n has n factors of the 1st degree (not necessarily unequal).

The proof that there are just n is simple (and familiar), when once it has been established that such an equation has necessarily a root (or roots). But none of the standard proofs of this latter fundamental theorem are simple.

(5) For Involution with fractional exponents we use the *definition* (See Ch. V., § 22).

$$z^{\pm m/n} = (z^{\pm m})^{+1/n}$$

This expression, by (1) and (2), gives n different values. Hence the different fractional expressions of a given fractional number are *not* equivalent in their most general use as exponents—though not, of course, giving altogether different results.

For this reason it is necessary to restrict the definition of z^f, for a given fractional exponent f, to mean what is given by the above relation when $\pm m/n$ is the *irreducible* fractional expression for f (*i.e.* when m, n are prime to one another).

The commuted form $(z^{+1/n})^{\pm m}$ may then be shown equivalent (it is not, apart from the restriction). And both forms give n values specifiable by

$$z^f = \sqrt[n]{r^{\pm m}}.\ \text{cis}\,(\pm m.\theta/n).$$

The "principal power" is that which comes from the principal amplitude, θ_0, of z, namely

$$[z^f] = \sqrt[n]{r^{\pm m}}.\ \text{cis}\,(\pm m.\theta_0/n).$$

(6) For Involution with irrational exponents the definition may then clearly be extended in the form

$$z^a = [r^a].\text{cis}\, a.\theta = [z^a].(+1)^a$$

if $[r^a]$ denote the positive real quantity given by § 24 and $[z^a]$, the "principal power," $= [r^a].\text{cis}\, a.\theta_0$.

This function of z, specified by z^a, has an infinity of values, corresponding to the infinity of values of the Amplitude, θ, of z.

(7) Thus, *for all real values of the exponent* a, we have

$$z^a = [r^a].\text{cis}\, a.\theta;\quad [z^a] = [r^a].\text{cis}\, a.\theta_0;$$

in which $[r^a]$ is as given by the restricted theory of §§ 22, 24.

The complication of the many-valuedness (the number of values being anything from 1 to ∞) is of less significance than might appear, because of the comparatively simple relation of all the other values to the principal value.

(ii.) The further, and final, development of the Number theory—to Involution with unreal exponents, and Logarithmation with both elements complex—turns on the theory of the "Exponential Function."

In this theory, which is neither elementary nor simple in detail, it is shewn (by one means or another*) that if

$$E(z) = 1 + z + z^2/2! + z^3/3! + \cdots \text{ ad inf.}$$

—an infinite series which is "convergent" for *all* values of z—then—

(1) When x is *real*, $E(x) = [e^x]$ and $E(\iota.x) = \text{cis}\, x$, if $e = E(1) = 1 + 1 + 1/2! + 1/3! + \cdots$ ad inf.—a positive real number (2·7182818285 . . . .) which may easily be proved irrational.

And the "Addition-theorem" (obvious in these particular cases) $E(z_1 + z_2) = E(z_1) \times E(z_2)$

is a *general* theorem; whence, in particular,

$$E(z) = E(x + \iota.y) = E(x).E(\iota.y) = [e^x].\text{cis}\, y$$

* There is a variety of modes of procedure. The writer has stated that which he prefers in a paper contributed to The Mathematical Gazette, October, 1906.

giving $[e^x]$, or $E(x)$, as modulus, and y as *an* Amplitude quantity, for the complex quantity specified by $E(z)$.

(2) The *inverse* functional expression, denoted by $L(z)$, such that if $w = L(z)$ then $z = E(w)$, may then be simply obtained in standard complex form. For, if $w = u + \iota.v$, we have $E(u). \text{cis}\, v = z$, and therefore

$$E(u) = [e^u] = |z| = r \quad \text{and} \quad \text{cis}\, v = \text{cis}\, \theta$$

and $u = \log_e r$, as defined (for positive real quantities) in § 23. Hence we have, for general specification of the L-function

$$L(z) = \log_e r + \iota.\theta = \log_e |z| + \iota.\, \text{Amp}\, z$$

specifying a function of z with, again, an infinity of values, corresponding to the infinity of values of θ. For its "principal value" we have

$$l(z) = \log_e r + \iota.\theta_0 = l(r) + \iota.\theta_0$$

and a variety of equivalents of $L(z)$ are given by

$$L(z) = l(|z|) + \iota.\text{Amp}\, z = l(z) + 2.k.\pi.\iota = L(|z|) + \iota.\text{amp}\, z$$

—$l(r)$ being an equivalent of the *real* $\log_e r$, as $E(x)$ is of $[e^x]$.

And $$L(z_1 \times z_2) = l(|z_1| \times |z_2|) + \iota.(\text{Amp}\, z_1 + \text{Amp}\, z_2)$$
$$= l(|z_1|) + l(|z_2|) + \iota.(\text{Amp}\, z_1 + \text{Amp}\, z_2)$$
$$= L(z_1) + L(z_2).$$

(3) These facts of (1) and (2) enable us to give the required extension of the last two operations. For, when a is real, we have $$z^a = [r^a].\text{cis}\, a.\theta.$$

And $$\log_e[r^a] = a. \log_e r^* = a.l(r),$$

therefore $$[r^a] = [e^{a.l(r)}] = E(a.l(r))$$

Hence $$z^a = E(a.l(r)).E(\iota.a.\theta)$$
$$= E(a.l(r) + \iota.a.\theta)$$
$$= E(a.(l(r) + \iota.\theta))$$
$$= E(a.L(z))$$

(4) Thus we arrive at the *general definition*

$$z^w = E(w.L(z)), \text{ when } z \text{ and } w \text{ are both complex,}$$
$$= E((u + \iota.v.).(l(r) + \iota.\theta))$$
$$= E(u.l(r) - v.\theta + \iota.(v.l(r) + u.\theta))$$
$$= E(u.l(r) - v.\theta).\, \text{cis}\,(v.l(r) + u.\theta)$$

giving this general involution in a standard complex form.

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* See Ch. V., § 24, (ii.), p. 45, footnote.

And $[z^w] = E(w.l(z))$

Cor. $L(z^w) = w.L(z)$.

(5) And for Logarithmation,

$$\log_z w = p \quad \text{if} \quad w = z^p = E(p.L(z))$$

therefore, if $L(w) = p.L(z)$.

Hence $\log_z w = L(w)/L(z) = (l(\rho) + \iota.\phi)/(l(r) + \iota.\theta)$, etc.

And $[\log_z w] = l(w)/l(z)$*

(6) The Involution and Logarithmation theorems, for the general case, are immediate corollaries :—

$$z^{p+q} = E((p+q).L(z))$$
$$= E(p.L(z)) \times E(q.L(z)) = z^p \times z^q$$
$$(z^p)^q = E(q.L(z^p)) = E(q.p.L(z)) = z^{p \times q}$$
$$(z_1 \times z_2)^p = E(p.L(z_1 \times z_2))$$
$$= E(p.(L(z_1) + L(z_2)))$$
$$= E(p.L(z_1)) \times E(p.L(z_2)) = z_1^p \times z_2^p$$

And

$$\log_z(p \times q) = L(p \times q)/L(z) = (L(p) + L(q))/L(z) = \log_z p + \log_z q$$
$$\log_z p^q = L(p^q)/L(z) = q.L(p)/L(z) = q.\log_z p$$
$$\log_p q = L(q)/L(p) = (L(q)/L(z))/(L(p)/L(z)) = \log_z q \log_z p$$

(iii.) The facts of §§ 26, 27 shew that we have arrived at a system of Numbers and of Operations on which there remain none of the restrictions which hampered free manipulation at the earlier stages. These are the general Numbers and Operations of Algebra.

Note on "VECTOR ANALYSIS."—

The "analysis" of Physical relationships, in terms of the Number-system, fails in one important respect. The numbers of Algebra are not adequate to the measurement of a *three*-dimensional system of vector-quantities (the practically important Vector case). This gives further emphasis to the fact, already noted (p. 53), that the relationship of vector-quantities (as such) to the number-system is relatively unimportant.

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* The notation $\mathrm{Log}_z w$, $\log_z w$ is sometimes used instead of $\log_z w$, $[\log_z w]$ respectively.

The basis of any "Vector Analysis" must necessarily be geometrical, because of the essential geometrical element —of Direction—in the specification of vector-quantities; and the purely geometrical "line-vector (or "length-vector") is, therefore, the basic type of vector. The fundamental operation of Addition, defined for line-vectors, (with its *unrestricted* Subtraction-inverse) is universally applicable to the Vector-quantities of Physics; and it is—as we have seen—in agreement also with Addition as defined for the Complex Numbers.

But there is no general development of Vector operations, closely analogous to that of the seven algebraic operations. The scheme of Vector Analysis which is practically most serviceable*—identified with the names of Heaviside and Willard Gibbs—turns (1) on the use of a type of vector which approaches one stage nearer to the number-system than the line-vector, (2) on the definition, for this type of vector, of a third operation—which has a partial analogy to Multiplication. The vector-quantities in question have *number*, instead of *length*, as corresponding "scalar" type, and are to be called "number-vectors" (or "pure vectors"). They may be regarded as "measures" of a system of vector-quantities of any other kind—in particular, of length-vectors; but *they are essentially vectors* (*i.e.*, *directed quantities*), *not numbers.*

The third operation may be specified in a simple geometrical way—by extending the rectangle-area definition of "product" of two lengths, to give the definition of "product" of two length-vectors OP, OQ as the *area-vector* of the parallelogram $OPSQ$, a vector of which the direction is a screw related direction at right angles to the directions of OP, OQ. This geometrical "product" is, of course, a quantity of different kind from its "factors"; but if we pass from the purely geometrical vectors, to number-vectors, by using measures of the two lengths and the measure of the area *with reference to the* "*square unit*" *of area*, we arrive at an appropriate *definition of the* "*product*" *of two number-vectors, as a third number-vector.* And this gives a Vector operation which is of great importance in Physical theory.

Thus we have *three operations* generally applicable to number-vectors, so as to give a *number-vector analysis*—which is the Vector Analysis of Heaviside and Gibbs. But the third operation is analogous to Multiplication in respect of only one of the three fundamental algebraic characteristics: it is subject (with Addition) to a Law of Distribution, but it is not subject to Laws of Commutation and Association. And there is (in consequence) no useful development analogous to that of the four remaining algebraic operations. Hence Vector Analysis is, as a mathematical theory, much more circumscribed than algebraic analysis; and its importance is, in fact, in great measure due to use in conjunction with the more fundamental analysis. And in this connection there is definable—as a corollary to the definition of the "vector product"—another "product," of two number-vectors, which is called their "*scalar product.*" It is a *number*, not a vector; and it also has an important place in Physical theory.

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* The theory of "Quaternions"—a quaternion being, not a quantity but, an operation— is more complicated.

APPENDICES

APPENDIX I.

PROOFS OF THE FUNDAMENTAL LAWS FOR THE NATURAL NUMBERS.

(i.) The Laws of Commutation and Association in Addition we take [See Ch. I., § 2] to be part of what we get directly from our most elementary experience of "counting." They are the mathematical expression of the facts as to the unique "sum" of a given set of natural numbers.

(ii.) The corresponding laws in Addition-and-Subtraction are deducible, as follows:—

(1) If, for example, $x = a - b + c - d - e + f - g$, all the symbols denoting natural numbers—restricted by the Natural Number condition on each subtraction, as it arises—then

$$x = y - g, \qquad \text{if } y = a - b + c - d - e + f$$

$$\therefore \quad x + g = y = z + f, \qquad \text{if } z = a - b + c - d - e.$$

Similarly $\quad z + e = a - b + c - d,$

and $\quad z + e + d = a - b + c = u + c$, say.

Hence $\quad (x + g) + (e + d) = (z + f) + (e + d)$;

$$\therefore \; x + g + e + d = (z + e + d) + f, \text{ by the Addition Laws (i.)}$$
$$= u + c + f.$$

But $\quad u = a - b, \quad \therefore u + b = a$; hence, in the same way,

$$x + g + e + d + b = (u + b) + c + f = a + c + f.$$

Thus $\quad x + \text{S} = \text{A} \quad$ or $\quad x = \text{A} - \text{S}$

if A denote the sum of the additive numbers (a, c, f) of the given expression for x, and S the sum of the subtractive numbers (b, d, e, g) of that expression: sums which are, of course [See (i.)], independent of the order of the additions from which they result.

The method is clearly of a *general* type, and the result in the form $x = \text{A} - \text{S}$ also general, subject only to the Subtraction conditions on an original expression for x of the type in question. Hence the Law of Commutation in Addition-and-Subtraction.

Thus, $a-b+c-d-e+f-g=c-g-d+a+f-e-b$ for numbers $a, \cdot \cdot \cdot g$, so restricted that every subtraction as it occurs is subject to the Natural Number condition.

Taking a numerical case,
$7-3-2+1+4-6=4-2+1+7-6-3\ (=12-11=1)$; but $1-6-2+4-3+7$ is, for example, *not* an admissible "commutation," in this case, because the subtraction $1-6$ does not satisfy the Natural Number condition.

(2) If, again, $x'=a-(b-c+d)-e+(f-g)$, for instance, and the natural numbers $a, \cdot \cdot \cdot, g$ conform to the Subtraction condition for all the subtractions involved, we may write

$$x'=a-p-e+q, \quad p=b-c+d, \quad q=f-g.$$

Then, by (1), $x'+p+e=a+q, \quad p+c=b+d, \quad q+g=f$; and, using the same principles as in (1),

$$x'+p+e+c+g=a+q+c+g$$

$$\therefore \quad x'+(p+c)+e+g=a+c+(q+g),$$

by the Addition Laws (i.)

$$\therefore \quad x'+b+d+e+g=a+c+f, \quad \text{or} \quad x'+\textsc{s}=\textsc{a}$$

Thus $\quad x'=\textsc{a}-\textsc{s}=x, \quad$ for the case in question.

The principles are clearly *general*, and give the Law of Association in Addition-and-Subtraction.

In the numerical case, we have

$$7-3-2+1+4-6=(7-3)-(2-1)+4-6, \text{ etc.};$$

but $7-(3+2-1)+(4-6)$ is, for example, *not* admissible, because it does not conform to the Subtraction condition.

(iii.) The Laws of Commutation and Association in Multiplication may be proved as follows :—

(1) $a.(b.c)=(c+c+\cdot\cdot\cdot b \text{ terms})+(\cdot\cdot\cdot)+\cdot\cdot+(\cdot\cdot\cdot)$,
there being a equal *composite* terms.
$=(c+c+\cdot\cdot\cdot a \text{ terms})+(\cdot\cdot\cdot)+\cdot\cdot+(\cdot\cdot\cdot)$,
there being b equal *composite* terms.
$=b.(a.c)$, using the primary propositions of (i.)

And, in particular, when $c=1, \quad a.b=b.a$

(2) Applying the results of (1),

$$a.(b.c) = a.(c.b) = c.(a.b) = c.(b.a)$$
$$= b.(a.c) = b.(c.a)\text{ ; and } (a.b).c = c.(a.b)$$

Hence all the ways of multiplying together *three* given natural numbers yield the same product—so proving the laws for that case.

(3) $a.b.c.d = a.(b.c.d)$, by definition
$$= a.(b.(c.d))$$
$$= a.b.(c.d) = (a.b).(c.d) = (a.b.c).d, \text{ by (2)}$$

(4) $a.b.c.d.e = a.(b.c.d.e) = a.b.(c.d.e)$, by definition
$$= (a.b).(c.d.e) = (a.b.c).(d.e)$$
$$= (a.b.c.d).e, \text{ by (2) and (3).}$$

(5) So, proceeding step by step, we arrive at the general result

$$a.b.c. \cdot \cdot \cdot l = a.(b.c. \cdot \cdot \cdot \cdot l) = (a.b).(c.d. \cdot \cdot \cdot l)$$
$$= (a.b.c).(d.e \cdot \cdot \cdot l) = \cdot \cdot \cdot \cdot$$

when the "association" is restricted to *two* groups of factors.

(6) The Law of Association then follows, by a converse use of (5), as in the following example:—

$$a.(b.c).(d.e.f.g).(h.k.l)$$
$$= a.(b.c).(\,(d.e.f.g).(h.k.l)), \text{ by defn.}$$
$$= a.(b.c).(d.e.f.g.h.k.l), \text{ by (5)}$$
$$= a.(b.c.d.e.f.g.h.k.l), \text{ by (5)}$$
$$= a.b.c.d. \cdot \cdot \cdot \cdot l.$$

The argument is clearly *general*.

(7) The Law of Commutation then follows, by "associating," then "commuting," then "disassociating," two consecutive factors—and repeating the process as often as necessary. For example,

$$c.e.a.d.b = c.(e.a).(d.b) = c.(a.e).(b.d)$$
$$= c.a.e.b.d = a.c.b.e.d, \text{ similarly}$$
$$= a.b.c.d.e, \text{ similarly.}$$

The argument is, again, clearly *general*.

(iv.) Since there are Laws for Multiplication exactly analogous to (if not nearly so obvious as) the Laws for Addition, and since Division has exactly the same general type of relation to Multiplication that Subtraction has to Addition, the Laws of Commutation and Association in Multiplication-and-Division may be proved by exactly the same type of argument as we used, in (ii.), for the Addition-and-Subtraction Laws. For example, if $x = a \div b \times c \div d \div e \times f \div g$, in which all the symbols denote natural numbers—restricted by the Natural Number condition on each division—we may write

$$x \times g = y = z \times f; \quad z \times e \times d = u \times c; \quad u \times b = a$$

giving $$x \times g \times e \times d \times b = a \times c \times f;$$

thus $$x \times \text{D} = \text{M}, \quad \text{or} \quad x = \text{M} \div \text{D}$$

if M denote the *product* of the multiplicative factors of the given expression for x, and D the *product* of its divisive factors; etc.

APPENDIX II.

THE SIGN OF MULTIPLICATION.

(i.) It has become a general practice to omit any specific sign of Multiplication, in *algebraic* forms—not, as we shall see, in the *arithmetical* forms which these "represent." The practice is to write ab for $a \times b$ or $a.b$; and the sole reason for this practice is economy—a principle, certainly, of great importance. If the economy is sound, the practice is sufficiently justified.

The form ab might, of course, have been used for any one of the operations—say, for Addition (and, in this connection, see (ii.), (2) below). The reason for attaching it to Multiplication is probably to be found in the elementary transition from "counting things" to "measuring quantities." We ordinarily *say* "five miles," and *write* "5 mls.," for what is more scientifically expressed as "five times the mile," written "5.ml"; and since the distinction between the two symbolic forms is commonly ignored, the mere combination of symbols comes naturally to be used as notation for a "product"—in particular, when the quantity multiplied is also a number.

The object of this note is to show reasons for regarding the practice in question as theoretically and practically unsound.

(ii.) The practice conflicts with the fundamental unity of Arithmetic and Algebra:

(1) If we write ab for the product, and then give to a and b natural number values, it is, of course, necessary to insert a specific sign of multiplication. Thus,

if $a = 2$ and $b = 3$, $ab = 2.3$ *not* 23;
if $a = 27$ and $b = 349$, $ab = 349 \times 27 = 27 \times 349$ *not* 27349.

This conflict of Algebraic practice with the Decimal Notation of Arithmetic is of very considerable psychological importance to the beginner, and contributes in no small measure (all the more, because subconsciously) to a prevalent haziness as to mathematical principles, on the part of those who have to use mathematics for a variety of practical purposes.

The basic importance of the Natural Numbers, to the Number-system, makes this point of the greater importance.

(2) The "mixed fraction" notation, while consistent with the Decimal Notation, brings out even more clearly the conflict with Algebra. For example, $5\frac{4}{9}$ means, *not* $5 \times 4/9$, but $5 + 4/9$.

(iii.) Since alphabetical symbols are used—not as such—in all algebraic work, there is an essential conflict of product-expressions with *words*.

(1) This has in itself some slight psychological significance for the beginner. It becomes of distinctly greater importance —to the non-mathematical—at the stage at which specifically mathematical words come into use: the most familiar of these being "log," "sin," "cos," etc. If the "functional" significance of these forms were more generally emphasised at an early stage, the complication would be more serious [See (iv.) below]; they are commonly regarded at first as mere contractions ("the logarithm of" a number; "the sine of" an angle; etc.). A good illustration is $\text{cis}\, x$ for $(\cos x + \iota . \sin x)$; only convention (a bad guide in scientific work) distinguishes this from a product form.

(2) The composite symbols characteristic of the Infinitesimal Calculus—δx (or $\triangle x$), δy, etc., dx, dy, etc., Dy, etc.—obviously bring the conflict of symbolism to an acute issue. It is quite unscientific to have any form (such as these) capable of two quite different interpretations, resolvable only by reference to the context. And there is nothing more true to the spirit of mathematical thought than the symbolism of the Calculus.

(3) Sir Napier Shaw has pointed out, more than once, the necessity—for the most modern of the sciences*—of a general

* The sciences of the Air.

development of such composite, or syllabic, symbolism, if only the conflict with product-forms could be avoided. In an article on "Symbolic Language of Science," in 'Nature,' Nov. 4, 1920, he points out the need to "expand . . . . notation . . . by proceeding from single letters to syllables . . . . an easy and effective way of dealing with the question *if we could do away with the convention that multiplication needs no symbol of operation and require that every operation should be represented by a suitable sign.*" (Italics supplied.)

(iv.) There is a further conflict with the notation for functional forms.

In $f(x)$ the presence of the brackets, otherwise unnecessary, is a sufficient indication of the meaning; but in $f(x+h)$ and $f(x-a)$, for example, there is, again, nothing but context—and experience—to distinguish the functional expression from the product. And this is the more important, in that the function-symbol, f, may often itself be conveniently used to denote the *quantity* specified by the expression $f(x)$.

(v.) To sum up: The mere juxta position of two symbols is not really adequate to the denoting of any specific operation. It has two many other important uses.

APPENDIX III.

NOUGHT AND INFINITY.

(i.) "Infinity" is a fundamental characteristic of the system of Natural Numbers. The term is used to express the fact that there is no end to the increasing sequence of these numbers.

The special symbol ∞, meaning infinity, is used in the expression of results obtained from increase without limit of a natural number variable, in certain types of general propositions. Thus (the symbols m, n denoting Natural Number variables)—

(1) $m+n=n+m>m$ gives $\infty+n=n+\infty=\infty$;
and, further, $\infty+\infty=\infty$.
Whence, inversely, $\infty-n=\infty$; while
$\infty-\infty$ *may have any value, up to* ∞ *itself.*

(2) $m\times n=n\times m \overset{=}{>} m$ gives $\infty\times n=n\times\infty=\infty$;
and, further, $\infty\times\infty=\infty$.
Whence, inversely, $\infty\div n=\infty$; while
$\infty\div\infty$ *may have any value, up to* ∞ *itself.*

Division *by* ∞ is otherwise irrelevant *within the Natural Number system*, which requires divisor not greater than dividend.

(3) $m^n \overset{=}{>} m$ gives $\infty^n=\infty$; and, further, $\infty^\infty=\infty$.
Whence, inversely, $\sqrt[n]{\infty}=\infty$; while
$\log_\infty\infty$ *may have any value, up to* ∞ *itself.*

(4) For the case of n^m, we note that
if $n>1$, $n^m=(1+p)^m>1+m.p>m$,*
Hence $n^\infty=\infty$, if $n>1$ [and, as in (3), $\infty^\infty=\infty$].
Whence, inversely, $\log_n\infty=\infty$, if $n>1$; while
$\sqrt[\infty]{\infty}$ *may have any value, from* 2 *to* ∞ *itself.*

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*See Appendix IV., (i.), (1).

But $\log_1\infty$ remains unprovided for; and 1^∞, we shall see [(iii.), (5), below], requires further discussion.

N.B.—*It is, of course, essential to these propositions that* ∞ *does not denote a number*, as all the other symbols (to which it is used analogously) do.

(ii.) The "number" nought (denoted by 0) we have seen to be a corollary to the Decimal Notation for the Natural Numbers. [See Ch. I., § 10].

(1) Its most obvious defining property is $n - n = 0$, regarded as the extreme case of Natural Number Subtraction—giving 0 the place before 1 in the Natural Number sequence.

From this relation we have, inversely, $n = 0 + n = n + 0$, and, again, $n - 0 = n$; and, further, $0 \pm 0 = 0$.

Also, $0 + \infty = \infty = \infty \pm 0$; and $\infty - \infty$ *may* have 0 as value. [See (i.), (1)].

(2) $0 \times n = 0 = n \times 0$—a product equal to the *lesser* of its own factors; further, $0 \times 0 = 0$.

Whence, inversely, $0 \div n = 0$, if $n > 0$; while

$0 \div 0$ *may have any value, from* 0 *to* ∞.

Also, $0 \div \infty = 0$*; but, inversely, the form $0 \times \infty$, we shall see [(iii.), (3), below], requires further discussion.

The general question of division by 0 also remains so far undetermined.

(3) $0^n = 0$, if $n > 0$; and, inversely, $\sqrt[n]{0} = 0$; while

$\log_0 0$ *may have any value from* 1 *to* ∞.

Also $\sqrt[\infty]{0} = 0$*, but, inversely, the form 0^∞, we shall see [(iii.), (4) below], requires further discussion.

(4) $n^0 = 1$, if $n > 0$, this being regarded as the extreme case [see (1), above] of n^{p-q}; and, inversely, $\log_n 1 = 0$, if $n > 0$.

[The evolution sign is never extended to $\sqrt[0]{}$. See § 22, (iv.)].

Also $\log_\infty 1 = 0$*; but, inversely, the form ∞^0, we shall see [(iii.), (5) below], requires further discussion.

(5) From (3) and (4) it is clear that 0^0 requires further discussion. [See (iii.), (5), below].

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* There is clearly no other possibility.

(iii.) In coming to apply these ideas to the general mathematical system, we note that the "real number" *nought* is no longer an extreme of the number-system—it is the number which separates the Negative Real Numbers from the Positive Real Numbers; that the system extends "to infinity" both positively and negatively; and that the extension to infinity (and through nought) is no longer by regular sequence (or "progression"), but by *continuous* increase or decrease.

The *results* of (i.) and (ii.) can be proved to hold generally when the natural number variable n is replaced by the real number variable x—with certain obvious modifications corresponding to the facts just stated. Thus*

(1) $(\pm\infty)\pm x = \pm\infty$; and, further, $x-(\pm\infty)=\mp\infty$;
$x\pm 0 = x$; " " , $0-x=(-x)$

(2) $(+\infty)\times x = \pm\infty = (+\infty)\div x$
and $(-\infty)\times x = \mp\infty = (-\infty)\div x$,
according as x is positive or negative;

$$0\times x = 0 = 0\div x, \text{ unless } x=0.$$

The proof in each case consists in showing that we can arrive at an algebraic sum, or product, further from 0 than any (great) number we care to name—or nearer to 0 than any (small) number we care to name—simply by giving a corresponding variation to one of the terms, or factors. We proceed to apply this type of argument to the two *new* cases, of division by ∞ and division by 0. [See (i.), (2) and (ii.), (2)].

(3) We use the elementary theorem that, if x, y be positive, the quotient x/y decreases or increases as y increases or decreases, respectively, while x is not varied

$$[x/y_1 - x/y_2 = x.(y_2-y_1)/(y_1 \cdot y_2)];$$

and the further fact that x/y may be made less than any positive real number, s, we care to name—*however small*—by sufficiently increasing y [$x/y < s$ if $y > x/s$]; or x/y made greater than any positive real number, G, we care to name—*however great*—by bringing y sufficiently near to 0 [$x/y > G$, if $y < x/G$].

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* It is important to have clearly in mind the two different meanings of the sign "+", and the *three* different meanings of "—" [See footnote to p. 17].

The mathematical expression of these facts, making obvious extension to negative values, is as follows :—

$$x \div (\pm \infty) = 0; \quad x \div 0 = \pm \infty.$$

This makes general provision for the zero-divisor.

And, inversely, $0 \times \infty$ *may have any real value.*

We further note that each of the forms $\infty \div \infty$ and $0 \times \infty$ is now reducible to the form $0 \div 0$—using the Real Number principle that division by a number is equivalent to multiplication by its reciprocal (and *vice versa*),and the facts (as above) that the reciprocals of 0 and ∞ are respectively ∞ and 0.

(4)* If $x > +1$, x^y increases to $+\infty$ with y. [See Appendix IV.]

$$\therefore \qquad x^{+\infty} = +\infty; \quad \text{and} \quad \infty^{\infty} = \infty.$$

If $0 < x < +1$, $x^y = (1/u)^y = 1/u^y$, where $u = 1/x > +1$; hence, by (3), $x^{+\infty} = 0$; and $0^{+\infty} = 0$†

Thus, again, $x^{-\infty} (= 1/x^{+\infty}) = 0$ or $+\infty$, according as $x > +1$ or $0 < x < +1$; and, inversely, $\log_x(+\infty) = \pm\infty$ and $\log_x 0 = \mp \infty$ in these respective cases. And $0^{-\infty} = +\infty$.

But the cases of $1^{\pm\infty}$, 0^0, ∞^0, $\log_0 0$, $\log_0 \infty$, $\log_\infty 0$, $\log_\infty \infty$ require further consideration.

(5) For the cases still outstanding we may use the Real Number identity [See Ch. V., § 23, (ii)]

$$\log_x y \equiv \log_a y \div \log_a x \equiv 1 \div \log_y x,$$

in which a denotes any *given* (positive real) number. And we may, for convenience, take a such that $a > +1$.

Thus, using (2), (3), (4), we have

$\log_0 x = 0$, since $\log_x 0 = \pm\infty$; and, inversely, 0^0 *any value*

$\log_1 x = \pm\infty$, since $\log_x 1 = 0$; and, inversely, $1^{\pm\infty}$ *any value*

$\log_\infty x = 0$ since $\log_x \infty = \pm\infty$; and, inversely, ∞^0 *any value.*

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* From this point we make the restrictions required for purely *real* number theory [See Ch. V.,§§ 21–4]. This implies a certain, comparatively unimportant, limitation of the results (which is not, however, final).

† This is clearly the only possibility.

Further,

$\log_0 0$ *any value* ; $\log_1 0 = \pm\infty$; $\log_\infty 0$ *any value** ;
$\log_0 1 = 0$; $\log_1 1$ *any value* ; $\log_\infty 1 = 0$;
$\log_0 \infty$ *any value** ; $\log_1 \infty = \pm\infty$; $\log_\infty \infty$ *any value†*.

(iv.) The several "*Indeterminate Forms*"—namely,

$$\infty - \infty\ ;\ 0\times\infty\ ;\ 0\div 0,\ \infty\div\infty\ ;\ 0^0,\ 1^{\pm\infty},\ \infty^0\ ;$$
$$\log_0 0,\ \log_1 1,\ \log_0\infty\ (\text{and } \log_\infty 0),\ \log_\infty\infty$$

—constitute the groundwork of the Infinitesimal Calculus theory of "Limits."

These forms, as has been indicated, are not independent of one another. All are, in fact, reducible to either of the two most important, namely, $0\div 0$ or $0\times\infty$—which are fundamental, respectively, to the Differential Calculus and to the Integral Calculus.

Expressions in a variable x which *fail* for some particular value of x, through assuming one or other of these forms *at* that value, generally yield what are called "limiting values"—to fill the blank so caused—determined by "continuity" [see Ch. V., § 20] with the values given by the expression in the immediate neighbourhood of the critical point. And it is such "limiting values" that constitute the subject-matter of the Infinitesimal Calculus.

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* This (real) form is necessarily negative.

† This (real) form is necessarily positive.

APPENDIX IV.

VARIATION OF FUNCTIONS SPECIFIED BY a^x AND $\log_a x$.

(i.) We require the following elementary inequalities :—*

(1) If $x > +1$ and $y > +1$, then $x.y > +1$.

For $x.y = (1 + u).(1 + v) = 1 + u + v + u.v$

in which u, v are positive.

Cor. 1. The product is greater than either factor.

Cor. 2. Similarly, $x.y.z. \cdots\cdots > 1 + (u + v + w + \cdots\cdots)$;

and $x^{+n} > 1 + n.u$ (for *positive integral exponents*).

(2) If $0 < x < +1$ and $0 < y < +1$, then $0 < x.y < +1$.

For $0 < x.y = x.(1 - v) = x - x.v < x < 1$

the quantity v being positive.

Cor. 1. The product is less than either factor.

Cor. 2. $0 < x^{+n} < +1$.

(3) If the product of two *different* real numbers $= +1$, one of the factors must be less, and the other greater, than $+1$.

Also, $\sqrt[n]{x} > +1$, if $x > +1$

and $0 < \sqrt[n]{x} < +1$, if $0 < x < +1$.

These follow immediately from (1) and (2).

(4) If $x > 0$ and $y > 0$, $x/y \lesseqgtr +1$ according as $x \lesseqgtr y$.

For $x/y - 1 = (x - y)/y$.

(ii.)†If $a > +1$, a^r increases, from 0 to $+\infty$, as r increases *by rational values* from $-\infty$ to $+\infty$:—

(1) If r is positive, $a^r > +1$; if r negative, $0 < a^r < +1$.

For, using $+m$, $+n$ as representative positive integral numbers,

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* These would be generally regarded as obvious facts ; but "obvious," in such a connection, means *very easy to prove* ; and the formal proofs are important.

† From this point the restrictions are to be observed which are characteristic of the Real Number theory of Involution and Logarithmation [See Ch. V., § 21·4].

$$a^{+m} > +1, \text{ by (i.), (1), Cor. 2}$$

and, therefore, $\sqrt[n]{a^{+m}} > +1$, by (i.), (3).

i. e., $a^{+m/n} > +1$.

And $a^{-m/n} = 1/a^{+m/n} < +1$, by (i.), (3)

(2) If $p < q$, then $a^p < a^q$.

For, $a^q/a^p = a^{q-p} > +1$, by (1), since $q > p$

Hence $a^q > a^p$, by (i.), (4).

(3) Now $a^{+\infty} = +\infty$, by (i.), (1), Cor. 2 [See Appendix III].

and $a^{-\infty} = 1/a^{+\infty} = 0$.

Hence, as r increases from $-\infty$, through 0, to $+\infty$, by rational values, a^r increases from 0, through $+1$, to $+\infty$.

(iii.) If $0 < a < +1$, we prove similarly (or deduce from (ii.) by substituting $a = 1/a'$) that as r increases from $-\infty$, through 0, to $+\infty$, by rational values, a^r *decreases* from $+\infty$, through $+1$, to 0.

(iv.) By means of these propositions of (ii.) and (iii.) we are enabled [See Ch. V., § 23, (i)] to define $\log_a b$ for all *positive real* values of a and b.

And it is an immediate consequence of the definition that, if $a > +1$, $\log_a x$ *increases*, from $-\infty$ to $+\infty$, as x *increases* "*continuously*"* from 0 to $+\infty$;
and, if $0 < a < +1$, $\log_a x$ *decreases*, from $+\infty$ to $-\infty$, as x *increases continuously* from 0 to $+\infty$.

(v.) Then again, from (iv.), we have, inversely, a^b specified for *positive* real values of a and *all* real values of b [See Ch. V., § 24.]

And it is, again, an immediate consequence that if $a > +1$, a^x *increases*, from 0 to $+\infty$, as x *increases continuously* from $-\infty$ to $+\infty$;
and, if $0 < a < +1$, a^x *decreases*, from $+\infty$ to 0, as x *increases continuously* from $-\infty$ to $+\infty$.

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* See Ch. V., § 20

(vi.) Further, the variations of $\log_a x$ and of a^x considered in (iv.) and (v.) are themselves continuous variations.

To prove this, it is sufficient—seeing that each of the variations in question is either increase throughout or decrease throughout—to show that $\log_a x$, in its variation, takes every real value, and that a^x, in its variation, takes every *positive* real value. And this is obviously true, since—

(1) $$\log_a x = v \quad \text{if} \quad x = a^v$$

which determines x, given v (real)—by (v.);

(2) $$a^x = p \quad \text{if} \quad x = \log_a p$$

which determines x, given p (real and positive)—by (iv.).

Thus, in the language of the Infinitesimal Calculus, $\log_a x$ and a^x specify " continuous functions of x," if a be "constant" with respect to x [See following Note]. These are important fundamental facts for Differential Calculus theory.

The graphs of $\log_a x$ and of a^x are clearly of the simplest possible (non-straight) type. Like all mutually inverse forms, one and the same *curve* can be used for both *graphs*—by interchange of " x-axis" and " y-axis" [See following Note]. The graphs should be " plotted out," by a sufficient number of guiding points, for several different values of the " constant" a—in particular, for $a = +10$ and $a = e = +2{\cdot}718 \cdot \cdot \cdot \cdot$ [See Ch. V., § 23 (ii), and Ch. VI., § 27 (ii), (1)], using Logarithm and Antilogarithm Tables.

Note on CONTINUOUS FUNCTIONS—

The relation $y = f(x)$ is said to specify y as a " *continuous function* of x," in so far as y varies continuously when x varies continuously.

The continuous variation, here under discussion, of the two (mutually inverse) functions specified by $\log_a x$ and a^x is of the simple type called " monotonic" (either *increase only* or *decrease only*). But continuous variation in general consists of alternate increase and decrease, separated by " turning-values" *at* which the variable ceases to increase and begins to decrease (" maximum values") or ceases to decrease and begins to increase (" minimum values").

Practically all the functions (of an " argument" x) which commonly occur are continuous functions of x, except *at* values of x which make y infinite.

From the graph of a function—drawn on tracing paper—the graph of the function "inverse" to it* may be obtained by tracing the figure through to the "back" of the sheet, and marking "Ox" the trace of Oy, and marking "Oy" the trace of Ox. Since "continuity" of a function corresponds to geometrical continuity of its graph, this makes it apparent that continuity of a function implies specific continuity of the inverse function.

A "constant," with respect to x, is to be thought of as a "variable," which is *not a function* of x: that is to say, "constant" is in contradistinction to "function," *not* to "variable."

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* $f(x)$, $\mathrm{F}(x)$ are said to specify "mutually inverse" functions of x, if $y = f(x)$ is equivalent to $x = \mathrm{F}(y)$ (and $y = \mathrm{F}(x)$ equivalent to $x = f(y)$). Thus x^3 and $\sqrt[3]{x}$, $\cos x$ and $\cos^{-1} x$, are examples.

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